GOVERNMENT ARTS AND SCIENCE COLLEG, KOVILPATTI – 628 503.
(AFFILIATED TO MANONMANIAM SUNDARANAR UNIVERSITY, TIRUNELVELI)
DEPARTMENT OF MATHEMATICS
STUDY E - MATERIAL
CLASS : II M.Sc. (MATHEMATICS) SEM: III
SUBJECT : MEASURE AND INTEGRATION(PMAM31)

MSU / 2016-17 / PG –Colleges / M.Sc.(Mathematics) / Semester -III / Ppr.no.9 / Core-7

Measure and Integration

Unit I : Lebesgue Measure – Lebesgue Outer Measure – The σ - Algebra of Lebesgue Measurable sets – Outer and Inner Approximation of Lebesgue Measurable sets – Countable Additivity, Continuity and the Borel – Cantelli Lemma – Lebesgue Measurable functions – Sums, Products and Compositions.
Chapter 2 : Sec 2.1 – 2.5 and **Chapter 3 :** Sec 3.1
Problems : Chapter 2 : 1 – 12, 16 – 18 and **Chapter 3** : 1 – 6

Unit II : Sequential pointwise Limits and Simple Approximation – Littlewood's Three Principles, Egoroff's Theorem and Lusins Theorem – Lebesgue Integration – The Riemann Integral – The Lebesgue Integral of a bounded Measurable function over a set of finite Measure – The Lebesgue Integral of a Measurable non – negative function – The general Lebesgue Integral – Countable Additivity and Continuity of Integration.
Chapter 3 : Sec 3.2 & 3.3 and **Chapter 4 :** Sec 4.1 – 4.5
Problems : Chapter 4 : 9 – 12, 16 – 20, 28, 30

Unit III : Differentiation and Integration – Continuity of monotone functions – Differentiability of monotone function : Lebesgue theorem – Functions of bounded variations : Jordan's theorem – Absolutely continuous functions – Integrating Derivatives : Differentiating Indefinite Integrals.
Chapter 6 : Sec : 6.1 – 6.5 (**No problems**)

Unit IV : Measure and Integration – Measures and Measurable sets – Signed Measures : The Hahn and Jordan Decompositions.
Chapter 17 : Sec : 17.1 – 17.4
Problems : Chapter 17 : 1, 2, 5, 13, 14, 18 & 19

Unit V : Integration over general Measure spaces : Measurable Functions – Integration of non – negative Measurable functions – Integration of general Measurable function (Upto the Lebesgue Dominated Convergence theorem only).
Chapter 18 : Sec : 18.1 – 18.3
Problems : Chapter 18 : 1, 2, 4, 5, 6, 19, 21, 31, 32

Text Book : Real Analysis, Fourth Edition, H.L.Royden, P.M.Fitzpatrick, PHI Learning Private Ltd.

Measure and Integration **(90 Hours)**

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Objective:

- Gain the knowledge of measure spaces and measure interruption
- Understanding the concept of lesbeague measure ,lesbeague integration and signed measure
- To provide the understanding of general measure spaces

Prerequisite:

- Basic knowledge of differentiation, integration and continuity of real functions

Outcome:

Knowledge gained about lesbeague theory and general measure spaces and their properties and construction.

Unit I: **LebesgueMeasure:**Lebesgue Measure – Lebesgue Outer Measure – The σ - Algebra of Lebesgue Measurable sets – Outer and Inner Approximation of Lebesgue Measurable sets – Countable Additivity, Continuity and the Borel – Cantelli Lemma.

Chapter 2 : Sec 2.1 – 2.5

Problems : Chapter 2 : 1 – 12 and 17

L 16

Unit II: **Lebesgue Measurable functions&Sequential pointwise Limits and related Theorems**: Lebesgue Measurable functions – Sums, Products and Compositions. Sequential pointwise Limits and Simple Approximation – Littlewood's Three Principles, Egoroff's Theorem and Lusin's Theorem

Chapter 3 : Sec 3.1 - 3.3 and

Problems :Chapter 3 : 1 – 3

L 19

Unit III: **LebesgueIntegration** :Lebesgue Integration – The Riemann Integral – The Lebesgue Integral of a bounded Measurable function over a set of finite Measure – The Lebesgue Integral of a Measurable non – negative function.

Chapter 4 : Sec 4.1 – 4.3

L 16

Unit IV: **Lebesgue Integral &Differentiablility**:The general Lebesgue Integral – Countable Additivity and Continuity of Integration. Differentiation and Integration – Continuity of monotone functions – Differentiability of monotone function: Lebesgue's theorem – Functions of bounded variations: Jordan's theorem.

Chapter 4 : Sec 4.4 & 4.5

Chapter 6 : Sec 6.1 - 6.3

L 19

Unit V: **Absolutely continuous functions &Signed Measures:** Absolutely continuous functions – Integrating Derivatives : Differentiating Indefinite Integrals. Measure and Integration – Measures and Measurable sets – Signed Measures : The Hahn and Jordan Decompositions – The Caratheodory measure induced by an outer measure – The construction of outer measure

Chapter 6 : Sec 6.4 & 6.5

Chapter 17 : Sec : 17.1 - 17.4

L 20

Text Book: **Real Analysis**, Fourth Edition, **H.L.Royden**, P.M.Fitzpatrick, PHI Learning Private Ltd.

Book for Reference:

Real Analysis Third Edition (PHI)-H.L.Royden Prentice hall of ofindia private limited –New Delhi (2006).

Unit - I.

Sec 2.1 :-

Introduction :-

The Riemann Integral of a bounded function over a closed, bounded interval is defined using approximations of the functions that are associated with partitions of its domains into finite collection of sub-intervals. The generalisation of Riemann integral to the Lebesgue integral will be achieved by using approximations of the functions that are associated with decomposition of its domain into finite collections of sets which we call Lebesgue measurable. (ie) Each interval is Lebesgue measurable.

The length $l(I)$ of an interval I is defined to be the difference of the end points of I if

I is bounded and infinity if I is unbounded. Length is an example of a set function (ie) a function that associates an extended real number to each set in a collection of sets.

The length of an open set will be the sum of the lengths of the countable number of open intervals of which it is composed. We construct a collection of sets called Lebesgue measurable and a set function of this collection called Lebesgue measure which is denoted by m. The collection of Lebesgue measurable sets is a σ-algebra which contains all open sets and all closed sets.

The set function m possesses the following three properties:-

1) The measure of an interval is its length. Every non-empty interval I is Lebesgue measurable and

$$m(I) = l(I)$$

(ii) Measure is translation invariant

If E is Lebesgue measurable and y is any member, then the translate of E by y $\textcircled{+}$ 1m

$$E+y = \{x+y \mid x \in E\}$$

is also Lebesgue measurable and

$$m(E+y) = m(E).$$

(iii) Measure is countably additive over countable disjoint union of sets.

If collection $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint collection of Lebesgue measurable sets, then

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k).$$

Remark:- It is not possible to construct a set function that possesses the above three properties and is defined for all sets of real numbers.

So, we first construct a set function called <u>outer measure</u> which we denote it by $\underline{m^*}$. It is defined for any set and in particular, for any interval.

The second stage is the construction of to determine what it means for a set to be Lebesgue measurable and show that the collection of Lebesgue measurable sets is a σ-algebra containing the open and closed sets. We then restrict the set function m^* to the collection of Lebesgue measurable sets denote it by m and prove m is countably additive. We call m by Lebesgue measure.

<u>Sec 2.2</u>:

<u>Outer Measure</u> - <u>Defn</u>:-

For a set A of real numbers, ~~for~~ consider the countable collections $\{I_k\}_{k=1}^{\infty}$ of non-empty open bounded intervals that cover A (ie) collections for which

$A \subseteq \bigcup_{k=1}^{\infty} I_k$

For each such collection consider the sum of the lengths of the intervals in the collection, we define the outer measure of A, $m^*(A)$ to be the infimum of all such sums.

(ie). $m^*(A) = \inf \left\{ \sum_{k=1}^{\infty} l(I_k) \,\middle|\, A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}$

Note: From the definition of outer measure (i) $m^*(\phi) = 0$.

(ii) Since any cover of a set B is also a cover of any subset of B, outer measure is monotone

If $A \subseteq B$; $m^*(A) \leq m^*(B)$.

Q2/ Thm : 1 A countable set has outer measure zero.

Proof: Let C be a countable set denoted as $C = \{c_k\}_{k=1}^{\infty}$

Let $\varepsilon > 0$.

For each natural no. k,

Define $I_k = \left(c_k - \frac{\varepsilon}{2^{k+1}}, c_k + \frac{\varepsilon}{2^{k+1}} \right)$

The countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ covers C.

$$m^*(C) \leq \sum_{k=1}^{\infty} l(I_k)$$

$$= \sum_{k=1}^{\infty} \left[c_k + \frac{\varepsilon}{2^{k+1}} - \left(c_k - \frac{\varepsilon}{2^{k+1}} \right) \right]$$

$$m^*(C) \leq \sum_{k=1}^{\infty} \left[\frac{c_k + \varepsilon - c_k + \varepsilon}{2^{k+1}} \right]$$

$$= \sum_{k=1}^{\infty} \left[\frac{2\varepsilon}{2^{k+1}} \right]$$

$$= \varepsilon \sum_{k=1}^{\infty} \frac{1}{2^k}$$

$\left[\frac{1}{2} + \frac{1}{2^2} + \cdots \right.$

$= \frac{1}{2} \left[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \right.$

$\left. = \frac{1}{2} \cdot \frac{1}{1 - 1/2} = 1 \right]$

$\therefore m^*(C) \leq \varepsilon$.

This inequality holds for each $\varepsilon > 0$

$$\therefore m^*(C) = 0.$$

Thm: 2 Outer measure of an interval is its lengths.

Proof:

Case (i) Let $[a,b]$ be a closed, bounded interval

Let $\varepsilon > 0$ be given.

Since the open interval $(a-\varepsilon, b+\varepsilon)$ contains $[a,b]$ we have $\cancel{m^*([a,b]) \le l(a-\varepsilon, b+\varepsilon)}$

$$m^*([a,b]) \le l(a-\varepsilon, b+\varepsilon)$$

$$= b+\varepsilon - a + \varepsilon$$

$$= b - a + 2\varepsilon.$$

this holds for every $\varepsilon > 0$

$$\therefore m^*([a,b]) \le b-a \longrightarrow (1)$$

It remains to show that

$$m^*([a,b]) \ge b-a.$$

This is equivalent to show that if $\{I_k\}_{k=1}^{\infty}$ is any countable collection of

open, bounded intervals covering $[a,b]$,
then $\sum_{k=1}^{\infty} l(I_k) \geq b-a$

By the Heine Borel theorem,
"Let F be a closed bounded set,
then for every open cover of F has
a finite subcover", any collection of
open intervals covering $[a,b]$ has
a finite subcollection that covers
$[a,b]$.

(ie) Choose any natural number
n for which $\{I_k\}_{k=1}^{n}$ covers $[a,b]$.

It is enough to show that

$$\sum_{k=1}^{n} l(I_k) \geq b-a \rightarrow (2)$$

Since a belongs to $\bigcup_{k=1}^{n} I_k$ there
must be one of the I_k's that
contains a

Select such an interval and
denote it by (a_1, b_1) we have
$a_1 < a < b_1$.

If $b_1 \geq b$, then the inequality (2) is satisfied since

$$\sum_{k=1}^{n} \ell(I_k) \geq b_1 - a_1 > b - a.$$

Otherwise $b_1 \in [a, b)$ and since $b_1 \notin (a_1, b_1)$, there is an interval in the collection $\{I_k\}_{k=1}^{n}$ which we label (a_2, b_2) distinct from (a_1, b_1) for which $b_1 \in (a_2, b_2)$.

(ii) $a_2 < b_1 < b_2$.

If $b_2 \geq b$, then the inequality (2) is satisfied since

$$\begin{aligned}\sum_{k=1}^{n} \ell(I_k) &\geq (b_1 - a_1) + (b_2 - a_2) \\ &= b_2 - (a_2 - b_1) - a_1 \\ &> b_2 - a_1 \\ &> b - a.\end{aligned}$$

We continue this selection process until it terminates as it must since there are only n intervals in the collection $\{I_k\}_{k=1}^{n}$. Thus we obtain a subcollection $\{(a_k, b_k)\}_{k=1}^{N}$ of

$\{I_k\}_{k=1}^{n}$ for which $a_1 < a$ and $a_{k+1} < b_k$ for $1 \le k \le N-1$. Since the selection terminated, $b_N > b$. $b_N > b$

Thus $\sum_{k=1}^{n} l(I_k) \geq$

$$\sum_{k=1}^{n} l(I_k) \geq \sum_{k=1}^{N} (a_k, b_k)$$

$$= (b_N - a_N) + (b_{N-1} - a_{N-1}) + \cdots + (b_1 - a_1)$$

$$= b_N - (a_N - b_{N-1}) - \cdots - (a_2 - b_1) - a_1$$

$$> b_N - a_1$$

$$> b - a$$

Thus the inequality ② holds.

Case (ii) :- If I is any bounded interval, then given $\varepsilon > 0$, there are two closed bounded intervals J_1 and J_2 such that $J_1 \subseteq I \subseteq J_2$

$$l(I) - \varepsilon < l(J_1)$$

$$\& \; l(J_2) < l(I) + \varepsilon.$$

By the inequality of outer measure and length for closed, bounded intervals and the monotonicity of outer measure $l(I) - \varepsilon < l(J_1) = m^*(J_1) \leq m^*(I) \leq m^*(I_2)$

$$l(I) - \varepsilon < l(J_1) = m^*(J_1) \leq m^*(I) \leq m^*(J_2) = l(J_2) < l(I) + \varepsilon$$

(ie) $l(I) - \varepsilon \leq m^*(I) \leq l(I) + \varepsilon$

$m^*(I) = l(I)$ for each $\varepsilon > 0$

Case (iii):

If I is an unbounded interval then for each natural number n, there is an interval $J \subseteq I$ such that $l(J) = n$.

Hence $m^*(I) \geq m^*(J) = l(J) = n$

This holds for each natural number n.

$\therefore m^*(I) = \infty$

AP. Thm:-3 Outer measure is translation invariant

(ie) For any set A, and a number y,

$m^*(A+y) = m^*(A)$.

Proof:- If $\{I_k\}_{k=1}^{\infty}$ is any countable collection of sets, then $\{I_k\}_{k=1}^{\infty}$ covers A iff $\{I_k+y\}_{k=1}^{\infty}$ covers $A+y$. If each I_k is an open interval, then each I_k+y is an open interval of same length and so

$$\sum_{k=1}^{\infty} l(I_k) = \sum_{k=1}^{\infty} l(I_k+y).$$

Taking infimum on both sides,

$$m^*(A) = m^*(A+y).$$

Thm:-4 Outer measure is countably sub additive.

(ie) if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets disjoint or not, then $m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} m^*(E_k)$

Proof:- If one of the E_k's has infinite outer measure, the inequality holds trivial.

Suppose each of the E_k's has finite outer measure.

Let $\varepsilon > 0$ be given.

For each natural number, k there is a countable collection $\{I_k\}_{k=1}^{\infty}$ of open bounded intervals for which $E_k \subseteq \bigcup_{i=1}^{\infty} I_{k,i}$ and $\sum_{i=1}^{\infty} l(I_{k,i}) \leq m^*(E_k) + \frac{\varepsilon}{2^k}$

$E_k \subseteq \bigcup_{i=1}^{\infty}$

Now, the collection $\{I_{k,i}\}_{1 \leq k,\, i < \infty}$ is a countable collection of open, bounded intervals that covers union $\bigcup_{k=1}^{\infty} E_k$, the collection is countable. Thus by the definition of outer measure,

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{1 \leq k,\, i < \infty} l(I_{k,i})$$

$$= \sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty} l(I_{k,i})\right]$$

$$< \sum_{k=1}^{\infty}\left(m^*(E_k) + \frac{\varepsilon}{2^k}\right)$$

$$= \sum_{k=1}^{\infty} m^*(E_k) + \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k}$$

$$= \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon \quad \left[\text{since } \sum_{k=1}^{\infty}\frac{1}{2^k} = 1\right]$$

Since this holds for every $\varepsilon > 0$, it holds for $\varepsilon = 0$.

If $\{E_k\}_{k=1}^{n}$ is any finite collection of sets disjoint or not, then

$$m^*\left(\bigcup_{k=1}^{n} E_k\right) \leq \sum_{k=1}^{n} m^*(E_k).$$

This finite subadditivity follows from countable subadditivity by taking $E_k = \phi$ for $k > n$.

Sec:- 2.3 The σ-algebra of Lebesgue measurable sets:-

Defn:-

A set E is said to be measurable provided for any set A,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

Note:- Since outer measure is fails to be countably additive and not even finitely additive, there are disjoint sets A & B for which

$m^*(A \cup B) < m^*(A) + m^*(B)$.

But if A is measurable, and B is any set disjoint from A, then

$$m^*(A \cup B) = m^*((A \cup B) \cap A) + m^*((A \cup B) \cap A^c)$$
$$= m^*(A) + m^*(B)$$

Thm:5 Any set of outer measure zero is measurable, in particular any countable set is measurable.

Proof: Let the E has outer measure zero.

Let A be any set.

Since $A \cap E \subseteq E$ and $A \cap E^c \subseteq A$,

By the monotonicity of outer measure,

$$m^*(A \cap E) \leq m^*(E) \text{ \& } m^*(A \cap E^c) \leq m^*(A)$$

$$m^*(A \cap E) \leq m^*(E) = 0$$

Thus $m^*(A) \geq m^*(A \cap E^c)$

$= 0 + m^*(A \cap E^c)$

$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c) \to (1)$

~~Since outer measure is finitely subadditive, and~~ $A = (A \cap E) \cup (A \cap E^c)$,

$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c) \to (2)$

From (1) & (2),

$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$

$\therefore$ E is measurable.

Thm:-6 The union of a finite collection of measurable set is measurable

Proof:- Show that Union of two measurable sets E_1 & E_2 is measurable

Let A be any set and E_1, E_2 are measurable sets.

Using the measurability of E_1, we have,

$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c)$.

Using measurability of E_2, we have,

$\cancel{m^*(A) = m^*(A \cap E_2) + m^*(A \cap E_2^c)}$.

$m^*(A) = m^*(A \cap E_1) + m^*((A \cap E_1^c) \cap E_2) + m^*((A \cap E_1^c) \cap E_2^c)$

We have the set identities,

$$(A \cap E_1^c) \cap E_2^c = A \cap (E_1 \cup E_2)^c$$

& $(A \cap E_1) \cup [A \cap E_1^c \cap E_2] = A \cap [E_1 \cup E_2)$

Using the above identities and finite subadditivity of outer measure,

$$m^*(A) = m^*(A \cap E_1) + m^*((A \cap E_1^c) \cap E_2) + m^*((A \cap E_1^c) \cap E_2^c)$$

$$m^*(A) = m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$$

Thus $E_1 \cup E_2$ is measurable.

Now, let $\{E_k\}_{k=1}^n$ be any finite collection of measurable sets.

To prove, the measurability of the union $\bigcup_{k=1}^n E_k$.

~~To pr~~ We will prove by induction on n.

The result is trivial for $n=1$.

Suppose, it is true for $n-1$.

Since $\bigcup_{k=1}^{n} E_k = \left(\bigcup_{k=1}^{n-1} E_k\right) \cup E_n$.

And the measurability of union of two measurable sets, the set $\bigcup_{k=1}^{n} E_k$ is measurable.

Theorem 7: Let A be any set collection $\{E_k\}_{k=1}^{n}$ a finite disjoint collection of measurable sets then

$$m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) = \sum_{k=1}^{n} m^*(A \cap E_k).$$

In particular, $m^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m^*(E_k)$

Proof: Proof is by induction on n.

For $n=1$, it is clearly true

Assume it is true for $n-1$.

Since the collection $\{E_k\}_{k=1}^{n}$ is disjoint

$$A \cap \left(\bigcup_{k=1}^{n} E_k\right) \cap E_n = A \cap E_n$$

$$A \cap \left(\bigcup_{k=1}^{n} E_k\right) \cap E_n^c = A \cap \left(\bigcup_{k=1}^{n-1} E_k\right)$$

Hence by the measurability of E_n and by the induction hypothesis,

$$m^*(A) = m^*(A \cap E_n) + m^*(A \cap E_n^c)$$

$$m^*\left(A\cap\left(\bigcup_{k=1}^{n}E_k\right)\right)=m^*(A\cap E_n)+m^*\left(A\cap\bigcup_{k=1}^{n-1}E_k\right)$$

$$=m^*(A\cap E_n)+\sum_{k=1}^{n-1}m^*(A\cap E_k)$$

$$=\sum_{k=1}^{n}m^*(A\cap E_k)$$

Remark: A set of real numbers is said to be a G_δ set provided it is the intersection of a countable collection of open sets and said to be an F_σ set provided it is union of countable collection of closed sets.

Theorem 8: The translate of measurable set is measurable.

Proof: Let E be a measurable set

Let A be any set and y be a real number.

By the measurability of E, and the translation invariant of outer measure

$$m^*(A) = m^*(A-y) = m^*[(A-y)\cap E] + m^*[(A-y)\cap E^c].$$

$\therefore E+y$ is measurable

$\therefore$ The translate of measurable set is measurable

Theorem: 9: The union of a countable collection of measurable set is measurable

Proof :- Let E be the union of a countable collection of measurable sets. (ie) $E = \bigcup_{k=1}^{\infty} E_k$.

Let A be any set. Let n be a natural number.

Define $F_n = \bigcup_{k=1}^{n} E_k$.

Since F_n is measurable, and $F_n^c \supseteq E^c$, $m^*(A) = m^*(A\cap F_n) + m^*(A\cap F_n^c)$

$$\geq m^*(A\cap F_n) + m^*(A\cap E^c)$$

By thm 7, $m^*(A \cap F_n) = \sum_{k=1}^{n} m^*(A \cap E_k)$

$\left[\because F_n = \bigcup_{k=1}^{n} E_k\right]$

Thus $m^*(A) \geq \sum_{k=1}^{n} m^*(A \cap E_k) + m^*(A \cap E^c)$

The Left Hand side of this inequality is independent of small n,

Thus $m^*(A) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k) + m^*(A \cap E^c)$

Hence by the countable subadditivity of outer measure

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$$

(ie) $E = \bigcup_{k=1}^{\infty} E_k$ is measurable.

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Theorem 10 Every interval is measurable

Proof:- To show that every interval is measurable, it is sufficies to show that every interval of the form (a, ∞) is measurable.

Consider such an interval, Let A be any set.

We assume a doesnot belong to A

We must show that

$$m^*(A) \geq m^*(A_1) + m^*(A_2) \text{ where } \rightarrow (1)$$

$$A_1 = A \cap (a, \infty) \quad , \quad A_2 = A \cap (a, \infty)^c$$

$$\text{(ie) } A_2 = A \cap (-\infty, a)$$

To verify (1), it is necessary and sufficient to show that for any countable collection, $\{I_k\}_{k=1}^{\infty}$ of open bounded intervals that covers A.

$$m^*(A_1) + m^*(A_2) \leq \sum_{k=1}^{\infty} l(I_k)$$

For such a covering, for each index k, define

$$I_k' = I_k \cap (-\infty, a)$$

$$I_k'' = I_k \cap (a, \infty)$$

Then I_k' and I_k'' are intervals and length of I_k $l(I_k) = l(I_k') + l(I_k'')$

Since collection $\{I_k'\}_{k=1}^{\infty}$ and $\{I_k''\}_{k=1}^{\infty}$ are countable collections of open bounded intervals that covers A_1 and A_2 respectively.

By the defn of outer measure.

$$m^*(A_1) \le \sum_{k=1}^{\infty} l(I_k')$$

$$m^*(A_2) \le \sum_{k=1}^{\infty} l(I_k'')$$

$$\therefore m^*(A_1) + m^*(A_2) \le \sum_{k=1}^{\infty} l(I_k') + \sum_{k=1}^{\infty} l(I_k'')$$

$$= \sum_{k=1}^{\infty} l(I_k') + l(I_k'')$$

$$= \sum_{k=1}^{\infty} l(I_k)$$

Sec 2.4:-

Outer and Inner approximations of Lebesgue measurable sets:-

Defn:- Excision Property:-

If A is a measurable set of finite outer measure that is contained in B, then $m^*(B \sim A) = m^*(B) - m^*(A)$

Theorem:- 11

Let E be any set of real numbers, then each of the following four assertions is equivalent to the measurability of E.

Outer approximations by open sets & G_δ sets)

(i) For each $\varepsilon > 0$, there is an open set O containing E for which $m^*(O \sim E) < \varepsilon$.

(ii) There is a G_δ set G containing E for which $m^*(G \sim E) = 0$

(Inner approximations by closed sets & F_σ sets:-)

(iii) For each $\varepsilon > 0$, there is a closed set F contained in E for which

$m^*(E \sim F) < \varepsilon$

(iv) There is a F_σ set F ~~containing~~ contained in E

for which $m^*(E \sim F) = 0$.

Proof: Assume E is measurable

To prove (i) Let $\varepsilon > 0$ be given.

case (i):- Consider the case that

$m^*(E) < \infty$.

By the defn of outer measure, there is a countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ which covers E and for which $\sum_{k=1}^{\infty} l(I_k) < m^*(E) + \varepsilon$.

Define $O = \bigcup_{k=1}^{\infty} I_k$, then O is an open set containing E.

By the defn of outer measure of O,

$$m^*(O) \leq \sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \varepsilon$$

$$m^*(O) - m^*(E) < \varepsilon .$$

Since E is measurable set and has finite outer measure by the excision property of measurable sets,

$$m^*(O \sim E) < \varepsilon = m^*(O) - m^*(E) < \varepsilon .$$

Case(ii): Consider $m^*(E) = \infty$

Then E may be expressed as a disjoint union of countable collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets, each of which has finite outer measure.

By case (i), for each index k, there is an open set O_k containing E_k for which $m^*(O_k \sim E_k) < \frac{\varepsilon}{2^k}$.

Then set $O = \bigcup_{k=1}^{\infty} O_k$ is open and it contains E.

And $O \sim E = \bigcup_{k=1}^{\infty} O_k \sim E \subseteq \bigcup_{k=1}^{\infty} (O_k \sim E_k)$

$$m^*(O \sim E) \leq \sum_{k=1}^{\infty} m^*(O_k \sim E_k)$$

$$< \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$$

$\therefore m^*(O \sim E) < \varepsilon$.

$\therefore$ The first property holds.

(ii) (i) $\Rightarrow$ (ii) For each natural no. k, choose an open set O that contains E for which $m^*(O_k \sim E) < \frac{1}{k}$

Define $G = \bigcap_{k=1}^{\infty} O_k$, then G is a G_δ set that contains E, since for each k, $G \sim E \subseteq O_k \sim E$.

By the monotonicity of outer measure,

$$m^*(G \sim E) \leq m^*(O_k \sim E) < \frac{1}{k}$$

$\therefore m^*(G \sim E) = 0$.

(iii) Assume property (ii).

To prove :- E is measurable.

Since a set of measure zero is measurable and the measurable sets are on σ-algebra, the set

$E = G \cap (G \sim E)^c$ is measurable.

Littlewood's first Principle :-

Second principle: Every measurable function is nearly continuous

Third: Every pointwise convergence seq of measurable function is nearly uniformly convergent

(Every measurable set is nearly a finite union of intervals).

Let E be a measurable set of finite outer measure. Then for each $\varepsilon > 0$ there is a finite disjoint collection of open intervals $\{I_k\}_{k=1}^{n}$ for which if $O = \bigcup_{k=1}^{n} I_k$ then

$m^*(E \sim O) + m^*(O \sim E) < \varepsilon$.

Proof:- By thm 11(i),

There is an open set U such that $E \subseteq U$ and $m^*(U \sim E) < \frac{\varepsilon}{2} \rightarrow (1)$.

Since E is measurable and has finite outer measure, from the excision property of outer measure U also has finite outer measure.

Every open set of real numbers

is the disjoint union of the countable collection of open intervals.

Let U be the union of the countable disjoint collection of open intervals $\{I_k\}_{k=1}^{\infty}$.

Each interval is measurable and its outer measure is its length

$$\therefore \sum_{k=1}^{n} l(I_k) = m^*\left(\bigcup_{k=1}^{n} I_k\right) \leq m^*(U) < \infty.$$

The right hand side of this inequality is independent of n.

$$\therefore \sum_{k=1}^{\infty} l(I_k) < \infty$$

Choose a natural number n for which $\sum_{k=n+1}^{\infty} l(I_k) < \frac{\varepsilon}{2}$.

Define $O = \bigcup_{k=1}^{n} I_k$.

Since $O \sim E \subseteq U \sim E$, by the monotonicity of outer measure,

$$m^*(O \sim E) \leq m^*(U \sim E) < \frac{\varepsilon}{2} \text{ (by ①)}.$$

→ ②.

Since $E \subseteq U$, $E \sim O \subseteq U \sim O$,

$$E \sim O \subseteq U \sim O = \bigcup_{k=n+1}^{\infty} I_k.$$

So that by the defn of outer measure

$$m^*(E \sim O) \leq \sum_{k=n+1}^{\infty} l(I_k) < \frac{\varepsilon}{2} \qquad (3)$$

By (2) & (3).

$$m^*(O \sim E) + m^*(E \sim O) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Remark:- By the defn of outer measure, for any bounded set E, regardless of whether or not is measurable, any $\varepsilon > 0$ there is an open set O, such that each subset of $E \subseteq O$, and $m^*(O) < m^*(E) < \varepsilon$ and

$$m^*(O) - m^*(E) < \varepsilon.$$

This doesnot imply that $m^*(O \sim E) < \varepsilon$, because the excision property

$$m^*(O \sim E) = m^*(O) - m^*(E)$$ is false unless E is measurable.

Sec 2.5:-

Countable additivity, continuity and Borel Cantelli lemma:

Defn:- The restriction of the set function outer measure to the class of measurable sets is called ~~leges~~ lebesgue measure.

It is denoted by m.

~~set~~ If E is a measurable set, its lebesgue measure m(E) is defined by

$m(E) = m^*(E)$

Theorem:-13 Lebesgue measure is countably additive:-

5M

(ie) If $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint collection of measurable sets then its union $\bigcup_{k=1}^{\infty} E_k$ is also measurable and ~~measure~~ of

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k).$$

Proof:

By thm 9, Union of countable collection of measurable set is measurable, $\bigcup_{k=1}^{\infty} E_k$ is measurable.

By thm 4, Outer measure is countably subadditive

$$\therefore m\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m(E_k) \rightarrow (1)$$

It remains to prove this inequality in the opposite direction.

By thm 7, each natural no. n,

$$m\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k)$$

Since $\bigcup_{k=1}^{\infty} E_k \supset \bigcup_{k=1}^{n} E_k$, by the monotonicity of outer measure,

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \geq m\left(\bigcup_{k=1}^{n} E_k\right)$$

$$= \sum_{k=1}^{n} m(E_k) \text{ for each } n.$$

The left hand side of this inequality is independent of n,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} m(E_k) \rightarrow (2)$$

From (1) & (2),

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k).$$

Theorem 14:- The set function Lebesgue measure defined on the σ-algebra of Lebesgue measurable sets assigns length to any interval, is translation invariant and is countable additive.

Defn:- A countable collection of sets, $\{E_k\}_{k=1}^{\infty}$ is said to be ascending provided, for each k, $E_k \subseteq E_{k+1}$, and said to be descending provided for each k,

$$E_{k+1} \subseteq E_k.$$

Theorem:-15 Continuity of measure:

Ap-16

The Lebesgue measure possess the following continuity properties.

(i) If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets, then

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k).$$

(ii) If $\{B_k\}_{k=1}^{\infty}$ is an descending collection of measurable sets, and $m(B_1) < \infty$, then

$$m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k).$$

Proof:

(i) Case (i):

$$m(A_k) = \infty.$$

If there is an index K_0 s.t for which $m(A_{K_0}) = \infty$.

Then by the monotonicity of measure, $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \infty$ $\forall k \geq K_0$.

$$\therefore m\left(\bigcup_{k=1}^{\infty} A_k\right) = \infty = \lim_{k\to\infty} m(A_k)$$

Case (ii): Consider $m(A_k) < \infty$, $\forall k$

Define $A_0 = \phi$ and define

$C_k = A_k \sim A_{k-1}$ for each $k \geq 1$.

Since the sequence $\{A_k\}_{k=1}^{\infty}$ is ascending $\{C_k\}_{k=1}^{\infty}$ is disjoint and

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} C_k.$$

By the countable additivity of m,

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = m\left(\bigcup_{k=1}^{\infty} C_k\right) = \sum_{k=1}^{\infty} m(C_k)$$

$$= \sum_{k=1}^{\infty} m(A_k \sim A_{k-1})$$

$\to (1)$

Since $\{A_k\}_{k=1}^{\infty}$ is ascending from the excision property of measure,

$$\sum_{k=1}^{\infty} m(A_k \sim A_{k-1}) = \sum_{k=1}^{\infty} [m(A_k) - m(A_{k-1})]$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} [m(A_k) - m(A_{k-1})]$$

$$= \lim_{n \to \infty} [m(A_n) - m(A_0)]$$

Since $m(A_0) = m(\phi) = 0$,

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m(A_k \sim A_{k-1})$$

$$= \lim_{n \to \infty} m(A_n).$$

Now,

To prove (ii), Define $D_k = B_1 \sim B_k$ for each k

Since the sequence $\{B_k\}_{k=1}^{\infty}$ is descending the sequence $\{D_k\}_{k=1}^{\infty}$ is ascending.

By part (i), $m\left(\bigcup_{k=1}^{\infty} D_k\right) = \lim_{k \to \infty} m(D_k)$

According to Demorgan's identities,

$$\bigcup_{k=1}^{\infty} D_k = \bigcup_{k=1}^{\infty} [B_1 \sim B_k] = B_1 \sim \bigcap_{k=1}^{\infty} B_k.$$

By the excision property of measure, for each k, since $m(B_1) < \infty$,

$$m(D_k) = m(B_1) - m(B_k)$$

∴ Measure of B_1.

$$\therefore m\left(B_1 \sim \bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty}\left[m(B_1) - m(B_k)\right]$$

$$m(B_1) \sim m\left(\bigcap_{k=1}^{\infty} B_k\right) = m(B_1) - \lim_{k\to\infty} m(B_k)$$

$$m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k)$$

<u>Defn:-</u> For a measurable set E, we say that a property holds <u>almost everywhere on E</u> or it holds for almost all <u>$x \in E$</u> provided there is a subset <u>E_0</u> of E for which <u>$m(E_0) = 0$</u> and the property holds for all $x \in E \sim E_0$.

<u>Theorem:-</u> <u>Borel - Catelli Lemma:-</u>

Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$, then almost all $x \in \mathbb{R}$ belong to atmost finitely many of the E_k's.

Proof:- For each n, by the countable subaddivity of m,

$$m\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \sum_{k=n}^{\infty} m(E_k) < \infty$$

Hence by the continuity of measure,

$$m\left(\bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right)\right) = \lim_{n\to\infty} m\left(\bigcup_{k=n}^{\infty} E_k\right)$$

$$= \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) = 0.$$

$\therefore$ Almost all $x \in \mathbb{R}$ fails to belong to $\bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right)$ and therefore belong to atmost finitely many E_k's.

<u>Unit - II</u>

3.1 Lebesgue measurable functions:

Lebesgue measurable function:

Here we consider extended real valued function.

<u>Proposition : 1</u> :-

The function f has measurable domain E, then the following statements are equal.

(i) for each real no. c, $\{x \in E \mid f(x) > c\}$ is measurable.

(ii) for each real no. c, $\{x \in E \mid f(x) \geq c\}$ is measurable

(iii) for each real no. c, $\{x \in E \mid f(x) < c\}$ is measurable

(iv) for each real no. c, $\{x \in E \mid f(x) \leq c\}$ is measurable.

Each of the properties implies for all extended real no. c.

$\{x \in E \mid f(x) = c\}$ is measurable.

Proof: (i) $\Rightarrow$ (ii).

$$\{x \in E \mid f(x) \geq c\} = \bigcap_{n=1}^{\infty} \{x \in E \mid f(x) > c - \tfrac{1}{n}\}$$

Since (i) is true, $\{x \in E \mid f(x) > c - \frac{1}{n}\}$ is measurable.

$\bigcap_{n=1}^{\infty} \{x \in E \mid f(x) > c - \frac{1}{n}\}$ is measurable (since m is a σ-algebra).

$\therefore \{x \in E \mid f(x) \geq c\}$ is measurable.

(ii) $\Rightarrow$ (i).

$$\{x \in E \mid f(x) > c\} = \bigcup_{n=1}^{\infty} \{x \in E \mid f(x) \geq c + \tfrac{1}{n}\}$$

Since (ii) is true, $\{x \in E \mid f(x) \geq c + \frac{1}{n}\}$ is measurable.

$\bigcup_{n=1}^{\infty} \{x \in E \mid f(x) \geq c + \frac{1}{n}\}$ is measurable

$\therefore \{x \in E \mid f(x) > c\}$ is measurable.

(i) $\Leftrightarrow$ (iv).

$$\{x \in E \mid f(x) > c\} = \mathbb{R} - \{x \in E \mid f(x) \leq c\} \rightarrow (1)$$

$$\{x \in E \mid f(x) \leq c\} = \mathbb{R} - \{x \in E \mid f(x) > c\} \rightarrow (2).$$

Since complement of measurable set is measurable.

Using (1) & (2), it follows that (i) $\Leftrightarrow$ (iv).

(ii) $\Leftrightarrow$ (iii).

$$\{x \in E \mid f(x) < c\} = R - \{x \in E \mid f(x) \geq c\} \rightarrow ⑤$$

$$\{x \in E \mid f(x) \geq c\} = R - \{x \in E \mid f(x) < c\} \rightarrow ⑥$$

Since complement of measurable set is measurable and by using ③ & ④. It follows that (ii) $\Leftrightarrow$ (iii).

(ii) $\Leftrightarrow$ (iii)

$\therefore \{x \in E / f(x) = c\}$ is measurable.

If $c = \infty$, then

$$\{x \in E / f(x) = \infty\} = \bigcap_{k=1}^{\infty} \{x \in E / f(x) > k\}$$

Since intersection of measurable set is measurable.

Defn:- An extended real valued function f defined on E, is said to be Lebesgue measurable or simply measurable provided its domain E is measurable and its satisfies one of the four statement of proposition 1.

Proposition - 2: Let the function f be defined on a measurable set E then f is measurable $\Leftrightarrow$ for each open set O, the inverse image of O under f, $f^{-1}(O) = \{x \in E \mid f(x) \in O\}$ is measurable.

Proof: Assume inverse image of each open set is measurable.

Since each interval (c, ∞) is open.

$$f^{-1}(c,\infty) = \{x \in E \mid f(x) \in (c,\infty)\}$$
$$= \{x \in E \mid f(x) > c\}$$

$f^{-1}(c,\infty) = \{x \in E \mid f(x) > c\}$ is measurable

$\therefore f$ is measurable

Conversely, Assume f is measurable.

Let $O = \bigcup_{k=1}^{\infty} I_k$, where I_k is a bounded open interval in $\mathbb{R}$.

$$I_k = (a_k, b_k)$$
$$= (a_k, \infty) \cap (-\infty, b_k)$$
$$= A_k \cap B_k \quad \forall k, \text{ where}$$

$A_k = (a_k, \infty)$

$B_k = (-\infty, b_k)$.

$$f^{-1}(A_k) = \{x \in E \mid f(x) \in A_k\}$$

$$= \{x \in E \mid f(x) \in (a_k, \infty)\}$$

$$= \{x \in E \mid f(x) > a_k\} \text{ is measurable}$$

$$f^{-1}(B_k) = \{x \in E \mid f(x) \in B_k\}$$

$$= \{x \in E \mid f(x) \in (-\infty, b_k)\}$$

$$f^{-1}(B_k) = \{x \in E \mid f(x) < b_k\} \text{ is } \underline{\text{measurable}}$$

Then $\bigcup_{k=1}^{\infty} \{f^{-1}(A_k) \cap f^{-1}(B_k)\}$ is measurable

(Since m is σ-algebra)

$$f^{-1}(O) = f^{-1}\left(\bigcup_{k=1}^{\infty} I_k\right)$$

$$= f^{-1}\left(\bigcup_{k=1}^{\infty}(A_k \cap B_k)\right)$$

$$= \bigcup_{k=1}^{\infty} f^{-1}(A_k \cap B_k)$$

$$f^{-1}(O) = \bigcup_{k=1}^{\infty}\left(f^{-1}(A_k) \cap f^{-1}(B_k)\right)$$

$f^{-1}(O)$ is measurable.

<u>Proposition-3</u> :- A continuous real valued function on the measurable domain is measurable.

<u>Proof</u> :- Let f be a continuous function on a measurable set E.

Let O be any open set.

Since f is continuous, $f^{-1}(O)$ is open in E

[Let f be a real-valued function defined on a set E of real numbers then f is continuous on E iff for each open set O, $f^{-1}(O) = E \cap U$ where U is open set].

$f^{-1}(O) = E \cap U$, where U is open in R

and so $U \in m$

Since E and U are measurable.

Then $E \cap U$ is measurable.

$f^{-1}(O)$ is measurable.

By proposition 2, f is measurable.

Note: A real valued function which is either increasing or decreasing is said to be monotone.

Proposition - 4:

A monotone function defined on interval is measurable

Proposition : 5

Let f be an extended real valued function on E.

(i) If f is measurable on E and

$f = g$ is almost everywhere on E then g is measurable on E.

(ii) For a measurable subset D of E, f is measurable on E iff the restriction of f to D and $E \sim D$ are measurable on E.

Proof:- Assume f is measurable on E

Define $A = \{x \in E \mid f(x) \neq g(x)\}$

Also $\{x \in E \mid g(x) > c\} = \{x \in A \mid g(x) > c\}$

$\cup \{x \in E \sim A \mid g(x) > c\}$

$= \{x \in A \mid g(x) > c\} \cup \{x \in E \sim A \mid f(x) > c\}$

(Since $f = g$ almost everywhere on $E \sim A$) $\to$ ①.

$A = \{x \in E \mid f(x) \neq g(x)\}$, $m(A) = 0$

Since $m(A) = 0$, A is measurable on E.

$\therefore \{x \in E \mid g(x) > c\} \subset A$

$m\{x \in E \mid g(x) > c\} \leq m(A) = 0$

$\therefore \{x \in A \mid g(x) > c\}$ is measurable on E

Since f is measurable on E, $\{x \in E \mid f(x) > c\}$ is measurable.

Since E and A are measurable on E, $E \sim A$ is also measurable.

From ①, $\{x \in E \mid g(x) > c\}$ is measurable on E.

Hence g is measurable.

(ii) For any c, $\{x \in E \mid f(x) > c\}$

$= \{x \in D \mid f(x) > c\} \cup \{x \in E \sim D \mid f(x) > c\} \rightarrow (1)$

Let $\{x \in D / f(x) > c\} = D \cap \{x \in E / f(x) > c\}$.

Since f is measurable on E and D is measurable $\{x \in D / f(x) > c\}$ is measurable

Similarly $\{x \in E \sim D / f(x) > c\} = E - D \cap \{x \in E / f(x) > c\}$

is measurable.

Then f/D, $f/E \sim D$ is measurable.

Conversely suppose f/D, $f/E \sim D$ are measurable

Then $\{x \in D / f(x) > c\}$ and $\{x \in E \sim D / f(x) > c\}$ are measurable

Now, by (1) $\{x \in E / f(x) > c\}$ is measurable.

$\therefore$ f is measurable.

Ap 16.

Linearity of measurable functions

Proposition : 6 Let f and g are measurable functions on E, that are finite almost everywhere on E. If (i) for any α, β, then $\alpha f + \beta g$ is measurable on E (linearity).

WD

(ii) fg is measurable (product).

Proof:- Assume f and g are finite a.e on E

Since $E_0 = \{x \in E \,/\, f(x), g(x) \neq \infty\}$ then $m(E \sim E_0) = 0$.

Since $\{x \in E \sim E_0 \,/\, (f+g)(x) > c\} \subset E \sim E_0$

$m(\{x \in E \sim E_0 \,/\, (f+g)(x) > c\}) \leq m(E \sim E_0) = 0$

Then $m(\{x \in E \sim E_0 \,/\, (f+g)(x) > c\}) = 0$ for all c.

$((f+g)/E \sim E_0)^{-1}(c, \infty)$ is measurable

$f+g/E \sim E_0$ is measurable.

Further $f+g/E_0$ is measurable.

By the previous proposition, $f+g$ is measurable on E.

From the above observation, we may assume that f and g are finite on all of E

If $\alpha = 0$, then αf is measurable.

Assume $\alpha \neq 0$ and observe that for real number c

$$\{x \in E \mid \alpha f(x) > c\} = \{x \in E \mid f(x) > c/\alpha\} \quad \text{if } \alpha > 0$$

$$\{x \in E \mid \alpha f(x) > c\} = \{x \in E \mid f(x) < c/\alpha\} \quad \text{if } \alpha < 0$$

Since f is measurable on E, we have αf is measurable on E.

To establish linearity consider the case that $\alpha = \beta = 1$.

For $x \in E$, $f(x) + g(x) < c$ then $f(x) < c - g(x)$, there is a rational number q for which $f(x) < q < c - g(x)$ as $\overline{\mathbb{Q}} = \mathbb{R}$.

$$\{x \in E \mid f(x) + g(x) < c\} = \bigcup_{q \in \mathbb{Q}} \left(\{x \in E \mid g(x) < c - q\} \cap \{x \in E \mid f(x) < q\} \right).$$

Since f and g are measurable on E, $\mathbb{Q}$ is countable and m is a σ-algebra we have $\{x \in E \mid f(x) + g(x) < c\}$ is measurable.

TP: fg is measurable on E.

W.K.T Let $fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$

Claim f^2 is measurable on E,

Let $\{x \in E / f^2(x) > c\}$

$c > 0$, $\{x \in E / f^2(x) > c\} = \{x \in E / f(x) < -\sqrt{c}\}$
$\cup \{x \in E / f(x) > \sqrt{c}\}$

$c < 0$, $\{x \in E / f^2(x) > c\} = E$

$(f+g)^2$ is measurable on E, f^2 & g^2 are measurable on E

Hence fg is measurable on E.

Proposition :- 7 Let g be a measurable real valued function on E. Let f be a continuous real valued function of all $\mathbb{R}$. Then composite maps $(f \circ g)$ is measurable on E.

(X) UA

(X) 5M

Proof: W.K.T by proposition (2)

A function is measurable iff the inverse image of each open set is measurable.

Let O be an open set,

$$(f\circ g)^{-1}(O) = g^{-1}(f^{-1}(O)) \rightarrow ①$$

Since f is continuous, the set $U = f^{-1}(O)$ is open.

Let g be measurable on E, (by propo-2)

$g^{-1}(U)$ is measurable on E.

From ①, $(f\circ g)^{-1}(O) = g^{-1}(U)$ is measurable on E.

$\therefore$ $f\circ g$ is measurable on E.

Proposition:-8:- For a finite family $\{f_k\}_{k=1}^{n}$ of measurable functions with common domain E, the function $\max\{f_1, f_2, \ldots, f_n\}$ and $\min\{f_1, f_2, \ldots, f_n\}$ are also measurable

Proof:- For any c, we have,

$$\{x \in E \mid \max\{f_1, f_2, \ldots, f_n\}(x) > c\}$$
$$= \bigcup_{k=1}^{n} \{x \in E \mid f_k(x) > c\}$$

So this set is measurable since it is the finite union of measurable sets.

Thus the function maximum of $\{f_1, f_2, \ldots f_n\}$ is measurable.

III^{ly} we can prove $\min\{f_1, f_2, \ldots, f_n\}$ is measurable.

Corollary: For a function f defined on E, we have the associated functions $|f|$, f^+ and f^- defined on E by

$$|f|(x) = \max\{f(x), -f(x)\}.$$

$$f^+(x) = \max\{f(x), 0\}$$

$$f^-(x) = \max\{-f(x), 0\}$$

3.2 Sequential Pointwise limits and simple approximation:

Defn: For a sequence $\{f_n\}$ of function with common domain E, a function f on E and subset A of E, we say that.

(i) A sequence $\{f_n\}$ converges to f pointwise on A, provided $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall\, x \in A$.

(ii) A sequence $\{f_n\}$ converge to f pointwise almost everywhere on A provided it converges pointwise on $A \sim B$ where $m(B) = 0$.

(iii) A sequence $\{f_n\}$ converges to f uniformly on A provided for each $\epsilon > 0$, there is an index N for which

$|f - f_n| < \varepsilon$ on A, $\forall n \geq N$.

Proposition: 9 Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise almost everywhere on E to the function f. Then f is measurable.

Proof: Given $\{f_n\} \to f$ pointwise a.e on E.

Let E_0 be the subset of E for which $m(E_0) = 0$ and $\{f_n\} \to f$ pointwise on $E \sim E_0$

Since $\{x \in E_0 / f(x) > c\} \subset E_0$ and $m(E_0) = 0$

$\Rightarrow m(\{x \in E_0 / f(x) > c\}) = 0$.

$(f|_{E_0})^{-1}(c, \infty)$ is measurable.

$f|_{E_0}$ is measurable.

By proposition 5, f is measurable iff $f|_{E \sim E_0}$ is measurable.

Replacing E by $E \sim E_0$, assuming that $\{f_n\} \to f$ pointwise on all E.

Fix a number c.

T.P: $\{x \in E / f(x) < c\}$ is measurable.

Observe that for $x \in E$,

Since $\lim_{n\to\infty} f_n(x) = f(x)$,

$f(x) < c$ iff there are natural numbers n and k for which $f_j(x) < c - \frac{1}{n}$ $\forall j \geq k$.

But any natural number n and j, since f_j is measurable

$\{x \in E \mid f_j(x) < c - \frac{1}{n}\}$ is measurable

For any k, the intersection of countable collection of measurable sets

$\bigcap_{j=k}^{\infty} \{x \in E \mid f_j(x) < c - \frac{1}{n}\}$ is measurable

Since, union of countable collection of measurable set is measurable

$$\{x \in E / f(x) < c\} = \bigcup_{1 \leq k, n < \infty} \left(\bigcap_{j=k}^{\infty} \{x \in E / f_j(x) < c - \frac{1}{n}\} \right) \text{ is measurable}$$

$\therefore \{x \in E / f(x) < c\}$ is measurable.

Hence f is measurable.

<u>Note:-</u> 1) Pointwise limit of continuous function may not be continuous.

2) Pointwise limit of Riemann Integral function may not be Riemann integrable.

Defⁿ: If A is any set The characteristic function of A, X_A is the function of R defined by

$$X_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Note:- X_A is measurable iff A is measurable.

Defⁿ:- A real valued function ϕ defined on a measurable set E is called simple provided it is measurable and it takes only a finite number of values

Note:- 1. A simple function only takes a real values.

2. The linear combination and product of simple functions are simple since each of them takes only finite number of values.

3. A simple of ϕ, has domain E and takes distinct values $c_1, c_2, \ldots c_n$ for

which

$$\varphi = \sum_{k=1}^{\infty} c_k \chi_{E_k} \text{ where}$$

$$E_k = \{x \in E \mid \varphi(x) = c_k\}$$

4. The expression of φ, the linear combination of characteristic function is called a canonical representation of simple function φ.

The simple approximation lemma :-

Let f be a measurable real valued function on E. Assume that f is bounded (ie) there is an $M \geq 0$ such that $|f| \leq M$. Then for each $\varepsilon > 0$, there are the simple functions ϕ_ε and ψ_ε ~~have~~ ~~the~~ defined on E have the following approximation properties.

$$\phi_\varepsilon \leq f \leq \psi_\varepsilon$$

and $0 \leq \psi_\varepsilon - \phi_\varepsilon < \varepsilon$ on E

Proof :- Let (c,d) be an open bounded interval which contains the image set of E,

(ii) $f(E) \subset (c,d)$.

Let $c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d$ be a partition of a closed, bounded interval $[c,d]$ such that $y_k - y_{k-1} < \varepsilon$ for all $1 \le k \le n$.

Define $I_k = [y_{k-1}, y_k)$ and

$E_k = f^{-1}(I_k)$ for $1 \le k \le n$.

Since for each k, each interval I_k is interval and f is measurable implies that each E_k is measurable.

Define $\phi_\varepsilon = \sum_{k=1}^{n} y_{k-1} \chi_{E_k}$ and

$$\psi_\varepsilon = \sum_{k=1}^{n} y_k \chi_{E_k}$$

Let $x \in E$,

since $f(E) \subset (c,d)$ there exists unique k, $1 \le k \le n$ for which $y_{k-1} \le f(x) < y_k$

$\Rightarrow \phi_\varepsilon(x) = y_{k-1} \le f(x) < y_k = \psi_\varepsilon(x)$

$\Rightarrow \phi_\varepsilon(x) \le f(x) \le \psi_\varepsilon(x)$ and

$\psi_\varepsilon(x) - \phi_\varepsilon(x) < \varepsilon$.

Hence the lemma.

(F) The simple approximation theorem

An extended real valued function f on E is measurable iff there exist a sequence $\{\phi_n\}$ of simple functions which converges pointwise on E, to f that has the property $|\phi_n| \leq |f|$ on E $\forall n$. If f is non-negative, then we may choose $\{\phi_n\}$ to be increasing.

Proof:- W.K.T simple functions are measurable

By hyp: Then $\{\phi_n\}$ is measurable

$\Rightarrow f$ is measurable.

Assume that f is measurable and $f \geq 0$.

Choose a natural number n.

Define $E_n = \{x \in E \,/\, f(x) \leq n\}$.

Clearly E_n is measurable

$\Rightarrow f/E_n$ is measurable.

We apply the simple approximation lemma to (f/E_n), for $\varepsilon = \frac{1}{n}$, there exist two simple functions ϕ_n & ψ_n on E_n having the following approximation properties.

$0 \leq \phi_n \leq f \leq \psi_n$ on E_n and $0 \leq \psi_n - \phi_n < \frac{1}{n}$ on E_n

Observe that $0 \leq \phi_n \leq f$ and $0 \leq f - \phi_n \leq \psi_n - \phi_n < \frac{1}{n}$ on E_n

~~$\Rightarrow 0 \leq f - \phi_n \leq \psi_n - \phi_n$ on E_n.~~

~~$\Rightarrow 0 \leq f - \phi_n < \frac{1}{n}$ on E_n~~

We extend ϕ_n to all of E, $\phi_n(x) = n$ if $f(x) > n$.

~~Let $x \in E$.~~

~~Case (i) Assume $f(x)$ is finite.~~

The function ϕ_n is a simple function defined on E and $0 \leq \phi_n \leq f \leq \psi_n$ on E.

Claim: $\{\phi_n\} \to f$ pointwise on E.

Let $x \in E$.

Case i) Assume $f(x)$ is finite

Choose a natural N such that $f(x) < N$.

$$0 \leq f(x) - \phi_n(x) < \frac{1}{n} \quad \forall n \geq N$$

$$\lim_{n \to \infty} \left(f(x) - \phi_n(x)\right) = 0$$

$$\Rightarrow f(x) - \lim_{n \to \infty} \phi_n(x) = 0$$

$$\Rightarrow \lim_{n \to \infty} \phi_n(x) = f(x)$$

Case (ii): Assume $f(x) = \infty$.

$\phi_n(x) = n$ for all n

$\lim_{n \to \infty} \phi_n(x) = \infty$
$= f(x)$

We choose each ϕ_n as the maximum of $\{\phi_1, \phi_2, \ldots \phi_n\}$, then $\{\phi_n\}$ be increasing.

Sec 3.3 Lemma:-

Assume E has finite measure. Let $\{f_n\}$ be the sequence of measurable functions on E that converges pointwise on E to the real valued function f. For each $\delta > 0$, $\eta > 0$, there is a measurable subset A of E and an index N such that $|f_n - f| < \eta$ on A for all $n \geq N$ and $m(E \sim A) < \delta$.

Proof:- For each k, the function $|f - f_k|$ is properly defined.

By proposition 9, f is measurable

Since f and f_k is measurable

$\therefore |f - f_k|$ is measurable

Let $E_n = \{x \in E \mid |f(x) - f_k(x)| < \eta\}$ for $k \geq n$

$$E_n = \bigcap_{k=n}^{\infty} \{x \in E \mid |f(x) - f_k(x)| < \eta\}.$$

Since the intersection of countable collection of measurable set is measurable

$\therefore E_n$ is measurable.

Also $\{E_n\}_{n=1}^{\infty}$ is the ascending collection of measurable sets and $E = \bigcup_{n=1}^{\infty} E_n$

By continuity of measure,

$$m(E) = \lim_{n\to\infty} m(E_n)$$

Since $m(E) < \infty$, we may choose an index N such that $m(E_N) > m(E) - \delta \rightarrow (*)$

Define $A = E_N$

$$m(E \sim A) = m(E) - m(A)$$
$$= m(E) - m(E_N)$$
$$< \delta$$

Apr 16 Write lemma

Thm: 5 Egoroff's theorem:-

(X) Assume E has finite measure. Let $\{f_n\}$ be the sequence of measurable functions on E that converges pointwise on E to the real valued function f. Then for each $\varepsilon > 0$, there is a closed set F contained in E for which $f_n \to f$ uniformly on F and $m(E \sim F)$.

Proof:- For each natural no. n, Let A_n be a measurable subset of E and $N(n)$ be the index which satisfy the preceeding lemma with $\eta = \frac{1}{n}$ and $\delta = \frac{\varepsilon}{2^{n+1}}$

(i) $m(E \sim A_n) < \frac{\varepsilon}{2^{n+1}} \to (1)$ and

$|f_k - f| < \frac{1}{n}$ for $k \geq N(n) \to (2)$

Define $A = \bigcap_{n=1}^{\infty} A_n$.

Since the intersection of countable collection of measurable set is measurable.

Hence A is measurable.

By De Morgan's Identity and countable subadditive of outer measure and (1)

$$m(E-A) = m(E - \bigcap_{n=1}^{\infty} A_n)$$

$$= m\left(\bigcup_{n=1}^{\infty} (E - A_n)\right)$$

$$\leq \sum_{n=1}^{\infty} m(E - A_n)$$

$$\leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} \cdot \frac{1}{2} \quad \left(\because \sum_{n=1}^{\infty} \frac{1}{2^n} = 1\right)$$

$$= \frac{\varepsilon}{2}$$

$$m(E - A) < \frac{\varepsilon}{2} \rightarrow (3).$$

We claim that $\{f_n\} \rightarrow f$ uniformly on A.

Let $\varepsilon > 0$, choose an index n_0 such that $\frac{1}{n_0} < \varepsilon$

Then (2) $|f_k - f| < \frac{1}{n_0}$ for $k \geq N(n_0)$ on A_{n_0}.

However, $A \subseteq A_{n_0}$ and $1/n_0 < \varepsilon$ and therefore

$|f_k - f| < \varepsilon$ for $k \geq N(n)$ on A.

Hence $\{f_n\} \rightarrow f$ uniformly on A and

$m(E - A) < \frac{\varepsilon}{2}$

Finally we choose a closed set F contained in A such that $m(A \sim F) < \frac{\varepsilon}{2}$ (by thm 11).

Now $E \sim F = (E \sim A) \cup (A \sim F)$

$m(E \sim F) = m(E \sim A) \cup m(A \sim F)$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$< \varepsilon$

$m(E \sim F) < \varepsilon$

Thus $\{f_n\} \to f$ uniformly on F and

$m(E \sim F) < \varepsilon$.

Note:- It is clear that Egoroff's theorem also holds if convergence pointwise almost everywhere on E and the limit function is finite almost everywhere.

theorem :- 6

Let f be a simple function defined on E, then for each $\varepsilon > 0$, there is a continuous function g on R and a closed set F contained in E for which $f = g$ on F and $m(E \sim F) < \varepsilon$.

Proof:- Let f be a simple function defined on E.

Let $a_1, a_2, \ldots, a_k$ be a finite number of real values taken by f, $1 \le k \le n$

Let $E_i = \{x \in E \mid f(x) = a_i\}$, $1 \le i \le n$.

As f is measurable, each E_i is also measurable since $\{a_k\}_{k=1}^{n}$ is distinct, and $\{E_k\}_{k=1}^{n}$ is disjoint.

By thm 11, For $\varepsilon > 0$, there exists a collection of closed sets $F_1, F_2, \dots, F_n$ contained in E for which

$$m(E_k \sim F_k) < \varepsilon/n \rightarrow (1) \quad \boxed{1 \le k \le n}$$

Define $F = \bigcup_{k=1}^{n} F_k$.

~~Finite~~ Since Finite union of closed sets is closed, F is closed.

$$m(E \sim F) = m\left(\bigcup_{k=1}^{n} E_k \sim \bigcup_{k=1}^{n} F_k\right)$$

$$= m\left(\bigcup_{k=1}^{n} (E_k \sim F_k)\right)$$

$$= \sum_{k=1}^{n} m(E_k - F_k)$$

$$< \sum_{k=1}^{n} \varepsilon/n = \frac{\varepsilon}{n} \cdot n = \varepsilon$$

$\therefore m(E \sim F) < \varepsilon$.

Define g on F and it takes

the values a_k on F_k, $1 \le k \le n$.

Since F_k's are disjoint, g is properly defined

Now we have to prove g is continuous on F.

Let $x \in F \Rightarrow x \in \bigcup_{k=1}^{n} F_k$

$\Rightarrow x \in F_i$ for some i, $1 \le i \le n$.

Then there is a open interval containing x which is disjoint from $\bigcup_{k \ne i} F_k$

Hence on intersection of this interval with F, the function g is constant.

Therefore, g is continuous on F.

By known result,

"Suppose f is a function that is continuous on a closed set F on real numbers then f has the continuous extension to all of R".

Thus g can be extended from a continuous function on a closed set

F to a continuous function on all of R

$\therefore g$ is continuous on R.

Clearly, the continuous function g on R satisfies $f = g$ on F.

Hence proved.

Theorem: 7

Lusin's theorem:-

Let f be real valued function defined on a measurable set E. Then for each $\epsilon > 0$, there is a continuous function g on R and a closed set $F \subset E$ for which $f = g$ on F and $m(E \sim F) < \epsilon$.

Proof: Consider $m(E) < \infty$

Let f be a real valued function defined on E,

By simple approximation thm,

there exists $\{f_n\}$ defined on E which converges pointwise on E to f.

Let n be any natural number.

f_n the above theorem 6, Replace f by

f_n and ε by $\varepsilon/2^{n+1}$ and we can choose a continuous function g_n on $\mathbb{R}$ and a closed set $F_n \subset E$ for which $f_n = g_n$ on F_n and $m(E \sim F_n) < \varepsilon/2^{n+1}$ $\to$ (1)

By Egoroff theorem,

for each $\varepsilon > 0$ and a closed set $F_0 \subset E$, then $\{f_n\} \to f$ uniformly on F_0 and $m(E \sim F_0) < \frac{\varepsilon}{2}$ $\to$ (2).

Define $F = \bigcap_{n=0}^{\infty} F_n$

$\therefore$ The set F is closed, since intersection of closed sets is closed.

By Demorgan's Identities and countable sub-additivity of measure,

Consider,

$$m(E \sim F) = m\left(E \sim \bigcap_{n=0}^{\infty} F_n\right)$$

$$= m\left(\bigcup_{n=0}^{\infty} (E \sim F_n)\right)$$

$$= m\left[(E \sim F_0) \cup \left(\bigcup_{n=1}^{\infty} (E \sim F_n)\right)\right]$$

$$\leq m(E \sim F_0) + m\left(\bigcup_{n=1}^{\infty} (E \sim F_n)\right)$$

$$< \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} m(E \sim F_n)$$

$$< \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} \varepsilon/2^{n+1}$$

$$= \varepsilon/2 + \frac{\varepsilon}{2} \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore m(E \sim F) < \varepsilon$.

Each f_n is continuous on F,

Since $F \subset F_n$ and $f_n = g_n$ on F_n.

Since $F \subset F_0$, $\{f_n\} \to f$ uniformly on F.

W.K.T.

"Uniform limit of a continuous function is continuous".

$\therefore f/_F$ is continuous on F.

By known result,

there is a continuous function g defined on all of $\mathbb{R}$ s.t $g/_F = f$ on F

(ie) $f = g$ on F.

Hence proved.

Unit - III

Lebesgue integration:-

The Riemann integral:-

Let f be a bounded real valued function defined on closed bounded interval $[a,b]$.

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition on $[a,b]$.

(ie) $a = x_0 < x_1 < \dots < x_n = b$

Define;

Lower Darboux sum $= L(f,P)$

$$= \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

Upper Darboux sum $= U(f,P)$

$$= \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

where $m_i = \inf\{f(x) \mid x_{i-1} < x < x_i\}$

$M_i = \sup\{f(x) \mid x_{i-1} < x < x_i\}$

We define,

Lower Riemann integral,

$(R) \underline{\int}_a^b f = \sup\{L(f,p) \mid p$ be a partition on $[a,b]\}$

Upper Riemann integral,

$(R) \overline{\int}_a^b f = \inf\{U(f,p) \mid p$ be a partition on $[a,b]\}$.

Since f is assumed to be bounded and the interval $[a,b]$ has finite length, the lower and upper Riemann integral are finite.

<u>Note:-</u> (i) The upper integral is always atleast as large as the lower integral.

(ie) $\underline{\int}_a^b f \leq \overline{\int}_a^b f$

(ii) If the two are equal we say that f is Riemann integrable over $[a,b]$.

(iii) The common value of Riemann integral of f over $[a,b]$ is denoted by $(R)\int_a^b f$

Defn:- Step function:

A real valued function ψ on $[a,b]$ is called a step function provided there is a partition $P=\{x_0, x_1, \dots, x_n\}$ of $[a,b]$ and the numbers $c_1, c_2, \dots, c_n$ such that for $1 \le i \le n$.

$$\psi(x) = c_i \text{ if } x_{i-1} < x < x_i$$

Observe that,

$$L(\psi, P) = \sum_{i=1}^{n} c_i (x_i - x_{i-1}) = U(\psi, P)$$

From the defn, we conclude that,

$$(R)\int_a^b \psi = \sum_{i=1}^{n} c_i (x_i - x_{i-1}).$$

We reformulate the defn of lower and upper Riemann integral as follows.

$$(R)\underline{\int_a^b} f = \sup\left\{(R)\int_a^b \phi \;\middle|\; \begin{array}{l}\psi \text{ a step function and}\\ \phi \le f \text{ on } [a,b]\end{array}\right\}$$

$$(R)\overline{\int_a^b} f = \inf\left\{(R)\int_a^b \psi \;\middle|\; \begin{array}{l}\psi \text{ a step function and}\\ \psi \ge f \text{ on } [a,b]\end{array}\right\}.$$

Example: [Dirichlet function]

Define f on $[0,1]$ by setting

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Let P be any partition on $[0,1]$.

$L(f, P) = 0$, $U(f, P) = 1$.

$(R)\underline{\int_0^1} f = 0 < 1 = (R)\overline{\int_0^1} f$.

So f is not Riemann integral.

The set of rational number on $[0,1]$ is countable

Let $\{q_k\}_{k=1}^{\infty}$ be an enumeration of the rational numbers on $[0,1]$.

For a rational number n, defined a function f_n on $[0,1]$, by setting

$$f_n(x) = \begin{cases} 1 & \text{if } x = q_k \text{ for some } k,\ 1 \le k \le n \\ 0 & \text{otherwise.} \end{cases}$$

$\Rightarrow$ Each f_n is a step function.

$\Rightarrow$ f_n is a Riemann integrable

~~$\Rightarrow$ $\{f_n\}$ is a Riemann integral function~~

$\Rightarrow$ $\{f_n\}$ is a increasing sequence of Riemann integrable function on $[0,1]$.

$\Rightarrow |f_n| \le 1$ on $[0,1]$ and $\{f_n\} \to f$

pointwise on $[0,1]$.

However the limit function ~~Hence~~ f fails on Riemann integrable on $[0,1]$.

Sec 4.2 Lebesgue integral of a bounded measurable function over a set of finite measure:-

A measurable real valued function ψ on a set E is said to be simple provided it takes only a finite number of real values. If ψ takes n distinct values of $a_1, a_2, \ldots a_n$, then by the measurability of ψ on E, $\psi^{-1}(a_i)$ is also measurable, then the canonical representation on E is given by

$$\psi = \sum_{i=1}^{n} a_i \chi_{E_i} \text{ where}$$

each $E_i = \psi^{-1}(a_i) = \{x \in E / \psi(x) = a_i\}$.

The canonical representation is characterised by each E_i's being disjoint and a_i's being distinct.

Defn: For a simple function ψ defined on a set of finite measure we define the integral of ψ over E by $\int_E \psi = \sum_{i=1}^{n} a_i\, m(E_i)$, where ψ has the canonical representation given by (1).

Lemma:- Let $\{E_i\}_{i=1}^{n}$ be a finite disjoint collection of a measurable subsets of a set of finite measure E. For $1 \le i \le n$, let a_i be a real number and if $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$, then $\int_E \phi = \sum_{i=1}^{n} a_i\, m(E_i)$.

Proof:- The collection $\{E_i\}_{i=1}^{n}$ is disjoint, the above ϕ may not be a canonical representation, since a_i's may not be distinct.

~~We must account for~~

Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the distinct values taken by ϕ.

~~Let~~ $a_j = \sum$

Let $A_j = \{x \in E \mid \phi(x) = \lambda_j\}$, $1 \le j \le m$

By defn of integral,

$$\int_E \phi = \sum_{j=1}^{m} \lambda_j m(A_j) \rightarrow ①$$

For $1 \le j \le m$, let I_j be the set of indices of i in $\{1, 2, \ldots, n\}$ for which $a_i = \lambda_j$,

Then $\{1, 2, \ldots, n\} = \bigcup_{j=1}^{m} I_j$, union is disjoint and $m(A_j) = m\left(\bigcup_{i \in I_j} E_i\right)$

(ie) $m(A_j) = \sum_{i \in I_j} m(E_i)$ $\quad 1 \le j \le m$

$$\sum_{i=1}^{n} a_i m(E_i) = \sum_{j=1}^{m} \left(\sum_{i \in I_j} a_i m(E_i) \right)$$

$$= \sum_{j=1}^{m} \lambda_j \left(\sum_{i \in I_j} m(E_i) \right)$$

$$= \sum_{j=1}^{m} \lambda_j m(A_j).$$

$$= \int_E \phi. \quad [\text{by } ①].$$

$$\sum_{i=1}^{n} a_i m(E_i) = \int_E \phi.$$

Proposition : Linearity and monotonicity of integration

118 Let φ and ψ be simple functions defined on a set of finite measure E. Then for any α and β

$$\int_E (\alpha\varphi + \beta\psi) = \alpha\int_E \varphi + \beta\int_E \psi$$

Moreover, if $\varphi \leq \psi$ on E, then

$$\int_E \varphi \leq \int_E \psi.$$

Proof: Since φ and ψ are simple functions, it takes only a finite number of real values.

Choose a finite disjoint collection $\{E_i\}_{i=1}^{n}$ measurable subsets of E, the union of which is E such that φ and ψ ~~let~~ are constant on each E_i

For each i, Let a_i and b_i be the values taken by φ and ψ respectively on E

Then by above lemma,

$$\int_E \varphi = \sum_{i=1}^{n} a_i \, m(E_i) \text{ and}$$

$$\int_E \psi = \sum_{i=1}^{n} b_i \, m(E_i)$$

The simple function $\alpha\phi + \beta\psi$ takes a constant value $\alpha a_i + \beta b_i$ on E_i

By above lemma,

$$\int_E (\alpha\phi + \beta\psi) = \sum_{i=1}^{n} (\alpha a_i + \beta b_i) \cdot m(E_i)$$

$$= \sum_{i=1}^{n} \alpha a_i \cdot m(E_i) + \sum_{i=1}^{n} \beta b_i \cdot m(E_i)$$

$$= \alpha \sum_{i=1}^{n} a_i\, m(E_i) + \beta \sum_{i=1}^{n} b_i\, m(E_i)$$

$$= \alpha \int_E \phi + \beta \int_E \psi.$$

To prove monotonicity,

Assume $\phi \leq \psi$ on E

Define $\eta = \psi - \phi$ on E

$$\int_E \psi - \int_E \phi = \int \psi - \phi = \int_E \eta \geq 0$$

Since non-negative simple function η has non-negative integral.

$$\therefore \int_E \psi \geq \int_E \phi.$$

Note 1: Since step function takes only a finite number of values and each interval is measurable, thus the step function is simple.

Note 2: Since The measure of singleton set is zero and the measure of an interval is its length, we infer from the linearity of Lebesgue integration of simple functions defined on a set of finite measure E that the Riemann integral over a closed, bounded, interval of step function agrees with the Lebesgue integral.

Defn: Let f be a bounded real valued function defined on a set of finite measure E, then we define lower and upper Lebesgue integral respectively as follows:-

$\sup\{\int_E \phi \mid \phi$ is simple and $\phi \leq f$ on $E\}$,

$\inf\{\int_E \psi \mid \psi$ is simple and $\psi \geq f$ on $E\}$

Defn: A bounded function f on a domain E of finite measure is said to be Lebesgue integrable over E provided its upper and lower Lebesgue integrals over E are equal. The common value of the upper and lower integrals is called the Lebesgue integral, or simply the integral of f over E and is denoted by $\int_E f$.

Thm: Let f be a bounded function defined on a closed bounded interval [a, b]. If f is Riemann integrable over [a, b]. Then it is Lebesgue integrable over [a, b] and the two integrals are equal.

Proof:- Suppose f is Riemann integrable,

Let $I = [a, b]$.

Let $R_l = \{(R)\int_I \phi \mid \phi \text{ is step function } \& \ \phi \leq f\}$

$L_l = \{\int_I \phi \mid \phi \text{ is simple function } \& \ \phi \leq f\}$

Since every step function is a simple function

$$R_\ell \subset L_\ell$$

$\Rightarrow \sup R_\ell \leq \sup L_\ell$

$$R_\upsilon = \left\{ (R)\int_I \psi \,/\, \psi \text{ is a step function } \& \ f \leq \psi \right\}.$$

$$L_\upsilon = \left\{ \int_I \psi \,/\, \psi \text{ is a simple function } \& \ f \leq \psi \right\}.$$

Also $R_\upsilon \subset L_\upsilon$

$\Rightarrow \inf R_\upsilon \geq \inf L_\upsilon$

W.K.T $\sup L_\ell \leq \inf L_\upsilon$

$$\sup R_\ell \leq \sup L_\ell \leq \inf L_\upsilon \leq \inf R_\upsilon.$$

Since f is Riemann integrable

$$\sup R_\ell = \inf R_\upsilon.$$

$\Rightarrow \sup L_\ell = \inf L_\upsilon.$

Thus f is Lebesgue integrable over $[a,b]$.

$$\sup R_\ell = \sup L_\ell = \inf L_\upsilon = \inf R_\upsilon$$

$$(R)\int_I f = \int_I f.$$

Hence the proof.

Eg:- The set E of rationals in $[0,1]$ is a measurable set of measure zero.

(ie) $E = \{\text{rational in } [0,1]\}$.

$m(E) = 0$ & E is measurable

The Dirichlet function f is restriction to $[0,1]$ of the characteristic function of E, χ_E.

(ie) $f = 1 . \chi_E$

Thus f is integrable over E.

$$\int_{[0,1]} f = \int_{[0,1]} 1 . \chi_E = 1 . m(E) = 0.$$

We have shown that f is not Riemann integrable over $[0,1]$.

Theorem: Let f be a bounded and measurable function on a set of finite measure E. Then f is a integrable over E.

Proof:- Let n be a natural number. By simple approximation lemma, with $\varepsilon = \frac{1}{n}$, there is a two simple functions ϕ_n and ψ_n defined on E then

$\phi_n \le f \le \psi_n$ on E and $0 \le \psi_n - \phi_n \le \frac{1}{n}$ on E.

By the monotonicity and linearity of integration for simple functions

$$0 \le \int_E \psi_n - \int_E \phi_n$$

$$= \int_E \psi_n - \phi_n$$

$$\le \int \frac{1}{n} = \frac{1}{n} m(E)$$

$$0 \le \int_E \psi_n - \int_E \phi_n \le \frac{1}{n} m(E) \rightarrow (1)$$

Moreover,

$$0 \le \inf\left\{ \int_E \psi \,\middle|\, \psi \text{ is a simple function on } E \text{ and } f \le \psi \right\} - \sup\left\{ \int \phi \,\middle|\, \phi \text{ is a simple function on } E \text{ and } \phi \le f \right\}.$$

$$0 \le \int_E \psi_n - \int_E \phi_n \le \frac{1}{n} m(E) \; [\text{by } (1)].$$

This is true for every n and $m(E) < \infty$

Then upper and lower integral are equal. Hence f is integrable over E

Hence proved.

Theorem:5 (Monotonicity and Linearity of integration):

Let f and g be bounded measurable function on a set of finite measure on E. Then for α and β

$$\int_E \alpha f + \beta g = \alpha \int_E f + \beta \int_E g$$

Moreover, if $f \leq g$ on E then $\int_E f \leq \int_E g$.

Proof: A linear combination of bounded measurable function is bounded and measurable.

By thm 4;

$\alpha f + \beta g$ is integrable

If ψ is a simple function, then $\alpha\psi$ is a simple function.

Conversely $\alpha \neq 0$, $\alpha\psi$ is simple function

$\Rightarrow \psi$ is simple function

For $\alpha > 0$, Since the Lebesgue integral is equal to upper Lebesgue integral.

$$\int_E \alpha f = \inf_{\alpha f \leq \psi} \int_E \psi$$

$$= \inf_{f \le \frac{\psi}{\alpha}} \int_E \frac{\psi}{\alpha} \cdot \alpha$$

$$= \alpha \inf_{f \le \frac{\psi}{\alpha}} \int \boxed{\frac{\psi}{\alpha}}$$

$$= \alpha \int_E f$$

$$\int_E \alpha f = \alpha \int_E f$$

For $\alpha < 0$, since Lebesgue integral is equal to upper and lower integral

$$\int_E \alpha f = \inf_{\alpha f \le \psi} \int_E \psi = \alpha \sup_{\boxed{\frac{\phi}{\alpha}} \le f} \int_E \frac{\phi}{\alpha}$$

$$= \inf_{f \ge \frac{\psi}{\alpha}} \int_E \frac{\psi}{\alpha} \cdot \alpha$$

$$= \alpha \cdot \sup_{f \ge \frac{\phi}{\alpha}} \int_E \frac{\psi}{\alpha}$$

$$= \alpha \cdot \int_E f$$

$$\int_E \alpha f = \alpha \int_E f$$

It is enough to prove that $\alpha = \beta = 1$

Let ψ_1 and ψ_2 be two simple functions for which $f \le \psi_1$ and

$g \leq \psi_2$ on E.

Then $\psi_1 + \psi_2$ is also simple function on E.

$f+g \leq \psi_1 + \psi_2$ on E.

Since $f+g$ is integrable and since $\int f+g$ is equal to upper Lebesgue integral of $f+g$ on E.

$$\int_E f+g \leq \int_E \psi_1 + \psi_2 = \int_E \psi_1 + \int_E \psi_2$$

The greatest lower bound for the sums of the integrals on the RHS as ψ_1 and ψ_2 vary among simple function for which $f \leq \psi_1$ and $g \leq \psi_2$ on E equals $\int_E f + \int_E g$

$$\int_E f+g \leq \int_E f + \int_E g \rightarrow (1).$$

Let ϕ_1 and ϕ_2 be two simple functions for which $\phi_1 \leq f$ and $\phi_2 \leq g$ on E

Then $\phi_1 + \phi_2$ is also simple function on E.

$\therefore \phi_1 + \phi_2 \leq f+g$ on E.

Since $f+g$ is integrable, and

$\int_E f+g =$ lower Lebesgue integral on E

$$\int_E f+g \geq \int_E \phi_1 + \phi_2 = \int_E \phi_1 + \int_E \phi_2$$

The least upper bound for the sums of the integrals on the R.H.S as ϕ_1 and ϕ_2 vary among simple functions $\phi_1 \leq f$ and $\phi_2 \leq g$ on E equals

$$\int_E f + \int_E g.$$

$$\int_E f+g \geq \int_E f + \int_E g \rightarrow (2).$$

From (1) & (2),

$$\int_E f+g = \int_E f + \int_E g$$

To prove monotonicity,

Assume $f \leq g$ on E.

Define $h = g - f$

$$\int_E g - \int_E f = \int_E g-f = \int_E h \geq 0.$$

Then h is non-negative.

$\therefore \psi \le h$ on E where $\psi \equiv 0$ on E

Since the Lebesgue integral of h is equal to lower Lebesgue integral

$$\int_E \psi \le \int_E h$$

$$\int_E h \ge \int_E \psi = 0$$

$$\int_E h \ge 0.$$

$$\int_E g - f \ge 0$$

$$\int_E g - \int_E f \ge 0$$

$$\int_E g \ge \int_E f$$

Hence $\int_E f \le \int_E g$

Corollary 6

Let f be a bounded measurable functions on a set of finite measure E. Suppose A and B are disjoint measurable subsets of E. Then

$$\int_{A\cup B} f = \int_A f + \int_B f.$$

Proof: Both $f.\chi_A$ and $f.\chi_B$ are bounded measurable functions on E. Since A and B are disjoint.

$f.\chi_{A\cup B} = f.\chi_A + f.\chi_B$. and.

$f.\chi_{A\cup B}(x) = 1$

Furthermore, for any measurable subsets E_1 of E,

$$\int_{E_1} f = \int_E f.\chi_{E_1}.$$

By linearity of integration,

$$\int_{A\cup B} f = \int_E f.\chi_{A\cup B}.$$

$$= \int_E f.\chi_A + \int_E f.\chi_B.$$

$$= \int_E f.\chi_A + \int_E f.\chi_B \text{ (by linearity)}$$

$$\int_{A\cup B} f = \int_A f + \int_B f$$

Corollary :- 1

Let f be a bounded measurable functions on a set of finite measure E. Then $\left|\int_E f\right| \leq \int_E |f|$.

Proof:- $|f|$ is bounded and measurable.

$$-|f| \leq f \leq |f|$$

By linearity and monotonicity of integration.

$$-\int_E |f| \leq \int_E f \leq \int_E |f|$$

$$\Rightarrow \left|\int_E f\right| \leq \int_E |f|.$$

Proposition 8: Let $\{f_n\}$ be a sequence of bounded measurable function on a set of finite measure E. If $\{f_n\}$ converges to f uniformly on E, then

$$\lim_{n\to\infty}\int_E f_n = \int_E f.$$

Proof: Given $\{f_n\} \to f$ uniformly on E

Now each function f_n is bounded $\to$ the limit function f is bounded.

Since the pointwise limit of a sequence of measurable function is measurable

$\therefore f$ is measurable.

Let $\varepsilon > 0$. Choose an index N for which $|f - f_n| < \frac{\varepsilon}{m(E)}$ on E for all $n \geq N$.

$$\left|\int_E f - \int_E f_n\right| = \left|\int_E f - f_n\right|$$

$$\leq \int_E |f - f_n|$$

$$\leq \int_E \frac{\varepsilon}{m(E)} \quad \forall\, n \geq N$$

$$\leq \frac{\varepsilon}{m(E)} \int_E 1.$$

$$\le \frac{\varepsilon}{m(E)} \cdot m(E).$$

$$\left|\int_E f - \int_E f_n\right| \le \varepsilon.$$

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Note:- $\lim_{n\to\infty} \int_E f_n = \int_E \lim_{n\to\infty} f_n = \int_E f$

This is known as the passage of the limit under the integral sign

Thm 6 Bounded convergence theorem.

Let $\{f_n\}$ be a sequence of measurable function on a set of finite measure E. Suppose $\{f_n\}$ is uniformly pointwise bounded on E, that is, there is a number $M \ge 0$ for which $|f_n| \le M$ on E $\forall$ n. If $\{f_n\} \to f$ pointwise on E, then $\lim_{n\to\infty} \int_E f_n = \int_E f$

Proof:- If the convergence is uniform the result follows from the above proposition

However Egoroff's theorem, tells us roughly that pointwise convergence is nearly uniform

The pointwise limit of a sequence of measurable function is measurable.

$\therefore f$ is measurable.

Clearly $|f| \le M$ on E $\forall n$ [Since $|f_n| \le M$ on E $\forall n$].

Let A be any measurable subset of E and n be a natural number.

By linearity and additive over domain of integral,

$$\int_E f_n - \int_E f =$$

$$\int_E f_n - \int_E f = \int_E (f_n - f)$$

$$= \int_A (f_n - f) + \int_{E \sim A} (f_n - f)$$

$$= \int_A (f_n - f) + \int_{E \sim A} f_n + \int_{E \sim A} (-f).$$

$$\left| \int_E f_n - \int_E f \right| \le \int_A |f_n - f| + \int_{E \sim A} |f_n| + \int_{E \sim A} |f|.$$

$$\leq \int_A |f_n - f| + \int_{E\sim A} M + \int_{E\sim A} M.$$

$$\left| \int_E f_n - \int_E f \right| \leq \int_A |f_n - f| + 2M \cdot m(E\sim A) \quad (1)$$

Let $\varepsilon > 0$, since ~~\#~~ $m(E)$ is finite and f is real valued function by Egoroff's theorem there is a measurable set A of E for which $\{f_n\} \to f$ uniformly on A & $m(E\sim A) < \frac{\varepsilon}{4M}$.

By uniform convergence, there is an index N for which $|f_n - f| < \frac{\varepsilon}{2(m(E))}$ on A for all $n \geq N$.

Therefore, for $n \geq N$, from (1) it follows ~~from~~ that

$$\left| \int_E f_n - \int_E f \right| \leq \frac{\varepsilon}{2(m(E))} \cdot m(A) + 2M \cdot \frac{\varepsilon}{4M}$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon.$$

$$\left| \int_E f_n - \int_E f \right| \leq \varepsilon.$$

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

The Lebesgue integral of non-negative measurable function:

Defn: A measurable function f defined on E is said to vanish outside a set of finite measure provided there is a subset E_0 of E for which $m(E_0) < \infty$ and $f \equiv 0$ on $E \sim E_0$.

It is convenient to say that a function that vanishes outside a set of finite measure has finite support and define its support to be $\{x \in E \mid f(x) \neq 0\}$.

If $m(E) = \infty$, if f is bounded and measurable on E but has finite support and we define the integral of f over E by

$$\int_E f = \int_{E_0} f \text{ where } E_0 \text{ has finite measure}$$

and $f \equiv 0$ on $E \sim E_0$

Defn: For f, a non-negative measurable function on E we define the integral of f over E by

$$\int_E f = \sup\left\{ \int_E h \,\Big|\, h \text{ bounded, measurable, finite support and } 0 \le h \le f \text{ on } E \right\}$$

Chebychev's inequality:

(7) Let f be a non-negative measurable function on E, then for any $\lambda > 0$

$$m\{x \in E \mid f(x) \geq \lambda\} \leq \frac{1}{\lambda} \int_E f \quad \rightarrow (1)$$

Proof: Let f be a non-negative measurable function on E.

Define $E_\lambda = \{x \in E \mid f(x) \geq \lambda\}$.

Case (i) Suppose $m(E_\lambda) = \infty$.

Let n be a natural number

Define $E_{\lambda,n} = E_\lambda \cap [-n, n]$

and $\psi_n = \lambda \cdot \chi_{E_{\lambda,n}}$.

Then ψ_n is bounded measurable function of finite support.

$\lambda \cdot m(E_{\lambda,n}) = \int_E \psi_n$ and $0 \le \psi_n \le f$ on E $\forall n$.

Also $E_\lambda = \bigcup_{n=1}^{\infty} E_{\lambda,n}$.

By continuity of measure,

$$\infty = \lambda \cdot m(E_\lambda) \leq \lambda \lim_{n\to\infty} m(E_{\lambda,n})$$

$$= \lim_{n\to\infty} \int_E \psi_n$$

$$\leq \int_E f$$

$$\lambda \cdot m(E_\lambda) \leq \int_E f$$

$$m(E_\lambda) \leq \frac{1}{\lambda} \int_E f$$

This inequality (1) holds since both sides equals ∞.

Case (ii): Suppose $m(E_\lambda) < \infty$.

Define $h = \lambda \chi_{E_\lambda}$.

Then h is bounded, measurable of finite support and $0 \leq h \leq f$ on E.

By defn of the integral of f over E

$$\lambda\, m(E_\lambda) = \int_E h \leq \int_E f$$

$$\lambda\, m(E_\lambda) \leq \int_E f$$

$$m(E_\lambda) \leq \frac{1}{\lambda} \int_E f$$

Hence the proof.

Proposition: 9 Let f be a non-negative measurable function then $\int_E f = 0$ iff $f = 0$ almost everywhere on E.

Proof: Assume $\int_E f = 0$.

By Chebyshev's inequality for each natural number n,

$$m\{x \in E / f(x) \geq \frac{1}{n}\} \leq n \int_E f = 0$$

$$m\{x \in E / f(x) \geq \frac{1}{n}\} \leq 0.$$

$$m\{x \in E / f(x) \geq \frac{1}{n}\} = 0$$

Consider $m\{x \in E / f(x) > 0\} = m\{\bigcup_{n=1}^{\infty} (x \in E / f(x) \geq \frac{1}{n})\}$

$$\leq \sum_{n=1}^{\infty} m\{x \in E / f(x) \geq \frac{1}{n}\}$$

$$= 0.$$

$$m\{x \in E / f(x) > 0\} = 0$$

Also given $f(x) \geq 0$

Conversely $f = 0$ a.e on E.

Let ϕ be a simple function & h be a bounded measurable function of finite support for which $0 \leq \phi \leq h \leq f$ on E.

Then $\phi = 0$ a.e on E [$\because f = 0$ a.e on

If $E_0 \subset E$, with $\phi \equiv 0$ on $E \sim E_0$ & $m(E_0) = 0$.

By previous example,

"Let E have measure zero. Let f be bounded function on E. Then f is measurable and $\int_E f = 0$"

$$\Rightarrow \int_{E_0} \phi = 0.$$

If $m(E) < \infty$ then $\int_E \phi = \int_{E_0} \phi + \int_{E \sim E_0} \phi = \int_{E \sim E_0} \phi$

If $m(E) = \infty$, then $\int_E \phi = \int_{E_0} \phi$.

Since this holds for ~~each~~ all ϕ, we have $\int_E h = 0$.

Since this is true for all h, we have $\int_E f = 0$

Hence the proof.

Fatou's Lemma :-

(ii) Let $\{f_n\}$ be a sequence of non-negative measurable function on E. If $\{f_n\} \to f$ pointwise a.e on E. Then $\int_E f \leq \liminf_{n\to\infty} \int_E f_n$.

Proof:- W.K.T if $E_0 \subset E$ and $m(E_0) = 0$

Then

$$\int_E f = \int_{E \sim E_0} f$$

Observe that pointwise convergente is on all E

Since f is a pointwise limit of a sequence of non-negative measurable on E, we have $f \geq 0$ and f is measurable

To verify in (1) it is necessary and sufficient to show that if h is any bounded measurable function of finite support for which $0 \leq h \leq f$ on E

$$\int_E h \leq \liminf \int_E f_n.$$

Let ~~l~~ h be such function on E

Choose $M \geq 0$ for $|h| \leq M$ on E

Define $E_0 = \{x \in E \mid h(x) \neq 0\}$

~~Then $m(E_0) < \infty$~~

~~Let n be a natural number~~

~~Define the function h_n on E~~

~~$h_n = \min\{f_n, h\}$.~~

Let n be a natural number

Define the function h_n on E.

$h_n = \min\{f_n, h\}$

Observe that h_n is measurable

Since $0 \le h_n \le M$ on E_0 and $h_n \equiv 0$ on $E \sim E_0$.

For each $x \in E$.

$h_n(x) = \min\{f_n(x), h(x)\}$.

$$\lim_{n\to\infty} \int_E h_n(x) = \min \lim_{n\to\infty} \int_E (f_n(x), h(x))$$

$$= \min(f(x), h(x))$$

$$= h(x).$$

$\{h_n(x)\} \to h(x)$

From the Bounded convergence theorem applied to the restriction to h_n of finite measure of E_0 on $E \sim E_0$

$$\lim_{n\to\infty} \int_E h_n = \lim_{n\to\infty} \int_{E_0} h_n = \lim_{n\to\infty} \int_{E_0} h_n$$

$$= \int h$$

However, every number n, $h_n \le f_n$ on E

By the defn of integral f_n on E

$$\int_E h_n \leq \int_E f_n$$

$$\int_E h = \lim_{n\to\infty} \int_E h_n$$

$$\leq \liminf \int_E f_n$$

This is true for $h_n \leq f_n$ on E.

Taking supremum LHS, $0 \leq h$ on E.

$0 \leq h \leq f$

$$\int_E f \leq \liminf \int_E f_n$$

We get the required result.

<u>Example:</u> Let $E = [0,1]$ and for a natural number n. Define $f_n = n \cdot \chi(0, 1/n)$. Then $\{f_n\}$ converges pointwise on E to $f \equiv 0$ on E.

However $\int_E f = 0 < 1 = \lim_{n\to\infty} \int_E f_n$

Let $E = \mathbb{R}$ and a natural number n. Define $g_n = \chi(n, n+1)$. Then $\{g_n\}$

convergence pointwise on E to g=0 on F

However $\int_E g = 0 < 1 = \lim_{n\to\infty} \int_E g_n$

Monotone convergence theorem:

Let $\{f_n\}$ be a increasing sequence of non-negative measurable function on E. If $\{f_n\} \to f$ pointwise a.e on E.

then $\lim_{n\to\infty} \int_E f_n = \int_E f$

Proof:- According to Fatou's lemma.

$\int_E f \leq \liminf_{n\to\infty} \int_E f_n \to (1)$.

However for a index number n $f_n \leq f$ a.e on E.

~~$\Rightarrow f_n \leq f$ a.e on E_0 and~~

~~$m(E \sim E_0) = 0$~~.

By monotonicity of integration of non-negative measurable on E $\int_E f_n \leq \int_E f$ on E

$\limsup_{n\to\infty} \int_E f_n \leq \int_E f \to (2)$.

From (1) & (2).

$\limsup_{n\to\infty} \int_E f_n \leq \int_E f \leq \liminf_{n\to\infty} \int_E f_n$

$\lim_{n\to\infty} \int_E f_n = \int_E f$

Defn: A non-negative measurable function f on a measurable set E is said to be integrable over E provided

$$\int_E f < \infty.$$

Proposition: Let ~~f be a~~ the non-negative function f be integrable over E. then f is finite a.e on E

Proof: Let n be natural number.

By Chebyshev's inequality

$$m\{x \in E / f(x) \geq n\} \leq \frac{1}{n} \int_E f \rightarrow ①$$

$$\{x \in E / f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \in E / f(x) \geq n\}$$

$$\subset \{x \in E / f(x) \geq n\} \forall n$$

By monotonicity of measure,

$$m\{x \in E / f(x) = \infty\} \leq m\{x \in E / f(x) \geq n\}$$

$$m\{x \in E / f(x) = \infty\} \leq \frac{1}{n} \int f \text{ (by ①)}$$

But $\int_E f$ is finite, hence

$$m\{x \in E / f(x) = \infty\} = 0$$

$\Rightarrow f$ is finite almost everywhere on E.

Beppo Levi Lemma:-

Let $\{f_n\}$ be an increasing sequence of non-negative measurable functions on E. If the sequence of integrals $\{\int_E f_n\}$ is bounded, then $\{f_n\}$ converges pointwise on E to a measurable function f that is finite almost everywhere on E and

$$\lim_{n\to\infty}\int_E f_n = \int_E f < \infty$$

Proof: W.K.T every monotonic sequence of extended real numbers converges to an extended real number.

Since $\{f_n\}$ is increasing sequence of extended real valued functions on E, we may define the extended real valued non-negative functions f pointwise on E.

By $f(x) = \lim_{n\to\infty} f_n(x)$ $\forall\, x \in E$.

Therefore, by monotone convergence theorem,

$$\left\{\int_E f_n\right\} \to \int_E f.$$

Therefore, since the sequence of real numbers $\left\{\int_E f_n\right\}$ is bounded, its limit is finite and so $\int_E f < \infty$.

By above proposition, f is finite almost everywhere on E.

Unit - IV

4.4. The General Lebesgue Integral.

For an extended real valued function f on E we have defined a positive part f^+ and negative part f^- on f. respectively by,

$$f^+(x) = \text{Max}\{f(x), 0\} \text{ and}$$
$$f^-(x) = \max\{-f(x), 0\} \;\forall\; x \in E.$$

Then f^+ and f^- are non-negative function on E.

$$f = f^+ - f^- \text{ on } E$$

and $|f| = f^+ + f^-$ on E

Observe that f is measurable iff both f^+ and f^- are measurable.

Thm: Let f be a measurable function on E. Then f^+ and f^- are integrable over E iff $|f|$ is integrable over E.

Proof: Assume that f^+ and f^- are ~~integ~~ non-negative functions on E

By the linearity of measurability integration for non-negative measurable functions,

$$|f| = f^+ + f^- \text{ on } E$$

$\therefore$ $|f|$ is integrable over E.

Conversely,

Suppose $|f|$ is integrable over E.

Since $0 \le f^+ \le |f|$ and $0 \le f^- \le |f|$ on E.

By the monotonicity of integration of non-negative function both f^+ and f^- are integrable over E.

$$\therefore \int_E f^+ \le \int |f| < \infty \text{ and } \int_E f^- \le \int_E |f| < \infty$$

f^+ and f^- are integrable over E.

Defn:- A measurable function f is said to be integrable over E provided $|f|$ is integrable over E.

$$\int_E f = \int_E f^+ - \int_E f^-$$

Proposition: 15

Let f be a integrable function on E. Then f is finite almost everywhere on E and $\int_E f = \int_{E \sim E_0} f$ if $E_0 \subseteq E$ and $m(E_0) = 0$.

Proof: Since f is integrable over E,

Then $|f|$ is integrable over E

$\Rightarrow |f|$ is finite almost everywhere on E

Since $0 \le f^+ \le |f|$ and $0 \le f^- \le |f|$

$\Rightarrow f^+$ and f^- are integrable over E

$\Rightarrow f^+$ and f^- are a.e on E.

$\Rightarrow f^+$ and f^- are almost everywhere on E

$\therefore f$ is finite almost everywhere on E

By monotonicity,

$$\int_E f^+ = \int_{E \sim E_0} f^+ \text{ and}$$

$$\int_E f^- = \int_{E \sim E_0} f^- \text{ if } E_0 \subset E, \text{ and } m(E_0) = 0.$$

$$\int_E f = \int_E f^+ - \int_E f^- = \int_{E \sim E_0} f^+ - \int_{E \sim E_0} f^-$$

$$\Rightarrow \int_E f = \int_{E \sim E_0} f \text{ where } m(E_0) = 0 \text{ if } E_0 \subseteq E.$$

Proposition: Integral comparison test

(ii) Let f be a measurable function on E. Suppose there is a non-negative function g that is integrable over E that dominates f in the sense $|\int_E f| \leq g$ on E. Then f is integrable over E and

$$\left|\int_E f\right| \leq \int_E |f|.$$

Proof: Given $|f| \leq g$

By monotonicity of non-negative measurable function $\int_E |f| \leq \int_E g < \infty$ [by defn of integrable]

$$\Rightarrow \int_E |f| < \infty$$

$\Rightarrow |f|$ is integrable over E

$\therefore f$ is integrable over E

$$\left|\int_E f\right| = \left|\int_E f^+ - \int_E f^-\right|. \quad \left|\int_E (f^+ - f^-)\right|$$

$$= \left|\int_E f^+ - \int_E f^-\right|. \text{ (by linearity)}$$

$$\int_E f \leq \int_E f^+ + \int f^- = \int_E |f|$$

$\cancel{\int_E (f^+ f^-) = \int_E |f|}$

$$\left| \int_E f \right| \le \int_E |f|$$

Linearity and monotonicity of integrable function:-

Let the function f and g be integrable over E. Then for any α, β, the function $\alpha f + \beta g$ is integrable over E and $\int_E \alpha f + \beta g = \alpha \int_E f + \beta \int_E g$. Moreover, if $f \le g$ on E, then $\int_E f \le \int_E g$

Proof:-

If $\alpha > 0$, then $(\alpha f)^+ = \alpha f^+$ and $(\alpha f)^- = \alpha f^-$

If $\alpha < 0$, then $(\alpha f)^+ = (-\alpha) f^-$ and $(\alpha f)^- = (-\alpha) f^+$.

Case (i) $\beta = 0$

Now, f is integrable over E

$\Rightarrow |f|$ is integrable over E.

$|\alpha f| = |\alpha| |f|$. ~~$\alpha f$~~ is integrable over E

$[\because \alpha > 0 \ \& \ |f| \ge 0]$

$\Rightarrow \alpha f$ integrable over E.

$$\int_E \alpha f = \int_E (\alpha f)^+ - \int_E (\alpha f)^-$$

If $\alpha > 0$, $\int_E \alpha f = \int_E (\alpha f)^+ - \int_E (\alpha f)^-$

$$= \int_E \alpha f^+ - \int_E \alpha f^-$$

$$= \alpha \int_E f^+ - \alpha \int_E f^-$$ [by linearity of non-negative functions]

$$= \alpha \left[\int_E f^+ - \int_E f^- \right]$$

$$\therefore \int_E \alpha f = \alpha \int_E f$$

If $\alpha < 0$, $\int_E \alpha f = \int_E (\alpha f)^+ - \int_E (\alpha f)^-$

$$= \int_E (-\alpha) f^- - \int_E -\alpha f^+$$

$$= -\alpha \int_E f^- - (-\alpha) \int_E f^+$$

$$\int_E \alpha f = \alpha \int_E f^+ - \alpha \int_E f^-$$

$$= \alpha \left[\int_E f^+ - \int_E f^- \right] = \alpha \int_E f$$

Case (ii)

To prove the linearity for the case $\alpha=\beta=1$

Since the linearity is true for non-negative functions, $|f|+|g|$ is integrable over E.

Since $|f+g| \leq |f|+|g|$.

By integral comparison test, $f+g$ is integrable over E.

By proposition 15, $f+g$ is finite a.e on E

We may assume that f and g are finite on E.

To verify the linearity it is to show that,

$$\int_E (f+g)^+ - \int_E (f+g)^- = \int_E f^+ - \int_E f^- + \int_E g^+ - \int_E g^- \rightarrow (1)$$

Now, $(f+g)^+ - (f+g)^- = f+g = f^+ - f^- + g^+ - g^-$

$(f+g)^+ - (f+g)^- = f^+ - f^- + g^+ - g^-$

Since each of these function takes real values on E,

$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$.

By linearity of integration for non-negative functions,

$$\int_E (f+g)^+ + \int_E f^- + \int_E g^- = \int_E (f+g)^- + \int_E f^+ + \int_E g^+$$

Since f, g and $f+g$ are integrable over E, each of these six integrals are finite.

Rearranging the integrals, we get ①.

(ie) $$\int_E (f+g)^+ - \int_E (f+g)^- = \int_E f^+ - \int_E f^- + \int_E g^+ - \int_E g^-$$

To prove monotonicity

Assume f and g are finite on E. Define $h = g - f$ on E

Then h is properly defined non-negative measurable function on E.

By linearity of integration for integrable functions and monotonicity of integration for non-negative measurable functions on E,

$$\int_E g - \int_E f = \int_E g - h = \int_E h \geq 0$$

$$\therefore \int_E g - \int_E f \geq 0$$

Hence $\int_E f \le \int_E g$

Corollary: Additive over domains of integration:

Let f be integrable over E. Assume A and B are disjoint measurable subsets of E.

Then $\int_{A \cup B} f = \int_A f + \int_B f$

Proof: We know that $|f.\chi_A| \le |f|$ and $|f.\chi_B| \le |f|$ on E.

By the integral comparison test, the measurable functions $f.\chi_A$ and $f.\chi_B$ are integrable over E.

Since A and B are disjoint,

$f.\chi_{A \cup B} = f.\chi_A + f.\chi_B$ on $E \rightarrow (1)$

We know that for a bounded measurable function f on a set of finite measure E

$\int_A f = \int_E f.\chi_A \rightarrow (2)$, where A is measurable subset of E.

Let $f \ge 0$ and for any measurable subset C of E

$\int_E f.\chi_C = \sup\Big\{\int_E h_1 \Big| h_1$ is a bounded measurable function of finite support and $0 \le h_1 \le f.\chi_C$ on $E\Big\}$

$= \sup\Big\{\int_E h_1 \Big| h_1$ is a bounded measurable functions of finite support and $h_1 = h.\chi_C = \begin{cases} h & \text{on } C \text{ where } h \text{ is} \\ 0 & \text{on } E \sim C \end{cases}$ bounded and $0 \le h \le f$ on $C\Big\}$

$= \sup\Big\{\int h \Big| h$ is a bounded measurable function with finite support and $0 \le h \le f.\chi_C$ on $\Big\}$.

[since $\int_E h_1 = \int_A h.\chi_C = \int_C h$ and by (2)]

$= \int_C f$

If f is integrable over E,

$\int_C f = \int_C f^+ - \int_C f^- = \int_E f^+.\chi_C - \int_E f^-.\chi_C$ (as $f^+, f^- \ge 0$)

$\int_C f = \int_E f.\chi_C$

Now $\int_E f.\chi_{A\cup B} = \int_E f\chi_A + \int_E f\chi_B$ (by linearity of integration for general function

$$\therefore \int_{A\cup B} f = \int_A f + \int_B f$$

Theorem: The Lebesgue Dominated convergence theorem:-

Let $\{f_n\}$ be a sequence of measurable function on E. Suppose there is a measurable function g that is integrable over E and dominates $\{f_n\}$ on E in the sense that $|f_n| \leq g$ on E for all n. If $\{f_n\} \to f$ pointwise almost everywhere on E, then f is integrable over E and

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Proof:- given $|f_n| \leq g$ on E for all n.

Since $\{f_n\} \to f$ pointwise almost everywhere on E, $|f| \leq g$ almost everywhere on E.

Since g is integrable over E, by integral comparison test, f and each f_n is also integrable over E

$\Rightarrow f$ and f_n (for all n) are finite almost everywhere on E

Since f is finite almost everywhere

on E, there exits a measurable set A_0 such that $m(A_0)=0$ and since each f_i is integrable over E, f_i is finite almost everywhere on E, therefore there exits a measurable set A_i such that $m(A_i)=0$

$$\therefore m\left(\bigcup_{i=0}^{\infty} A_i\right) \leq \sum_{i=0}^{\infty} m(A_i)=0$$

$$\Rightarrow m\left(\bigcup_{i=0}^{\infty} A_i\right)=0.$$

$\therefore$ By excising a countable collection of sets of measure zero, we may assume that f and f_n are finite on E (by prop-15).

Clearly the function $g-f$ and $g-f_n$ ($\forall n$) are properly defined non-negative measurable functions on E.

Also, $\{g-f_n\}$ converges pointwise almost everywhere on E to $g-f$.

By Fatou's lemma,

$$\int_E g-f \leq \liminf \int_E g-f_n$$

By linearity of integration,

$$\int_E g - \int_E f = \int_E g-f \leq \liminf_{n\to\infty} \int_E g-f_n$$

$$= \int_E g - \lim_{n\to\infty} \sup \int_E f_n$$

$$\therefore -\int_E f \leq -\lim_{n\to\infty} \sup \int_E f_n$$

$$\lim \sup \int_E f_n \leq \int_E f \to ①$$

IIIrly,

By considering sequence $\{g + f_n\}$, we obtain $\lim\inf \int_E f_n \geq \int_E f \to ②$.

From ① & ②, we have,

$$\lim \sup \int_E f_n \leq \int_E f \leq \lim\inf \int_E f_n$$

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Sec 4.5 The countable additive and continuity of integration

Thm: 20 Let f be a integrable on E and $\{E_n\}_{n=1}^{\infty}$ a disjoint countable collection of measurable subset of E whose union is E. Then $\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f$

Proof: Let n be a natural number.

Define $f_n = f.\chi_n$, where χ_n be a characteristic function of measurable set $\bigcup_{k=1}^{n} E_k$.

$\Rightarrow f_n$ measurable function on E.

$\Rightarrow |f_n| \leq |f|$ on E.

Observe that $\{f_n\} \to f$ pointwise on E

By, Lebesgue dominated convergence theorem

$$\int_E f = \lim_{n\to\infty} \int_E f_n \longrightarrow (*)$$

Since $\{E_n\}_{n=1}^{\infty}$ is disjoint, by additivity over domain of integration and that for each n,

$$\int_E f_n = \sum_{k=1}^{n} \int_{E_k} f.$$

$(*) \Rightarrow \int_E f =$

$$\sum_{k=1}^{n} \int_{E_k} f = \int_{\bigcup_{k=1}^{n} E_k} f = \int_{E=\bigcup_{k=1}^{n} E_k} f.\chi_n = \int_E f_n$$

$$(*) \Rightarrow \int_E f = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \int_{E_k} f \right]$$

$$\int_E f = \sum_{k=1}^{\infty} \int_{E_k} f$$

$$\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f .$$

Theorem 2! <u>Continuity of integration</u>:

Let f be integrable on E,

(i) If $\{E_n\}_{n=1}^{\infty}$ is an ascending countable collection of measurable subsets of E, then

$$\int_{\bigcup_{n=1}^{\infty} E_n} f = \lim_{n\to\infty} \int_{E_n} f$$

(ii) If $\{E_n\}_{n=1}^{\infty}$ is a descending countable collection of measurable subsets of E,

Then $\int_{\bigcap_{n=1}^{\infty} E_n} f = \lim_{n\to\infty} \int_{E_n} f$.

Sec 6.1 <u>Differentiation and integration</u>

<u>Continuity of monotonic function</u>:

Thm: Let f be monotonic function on open interval (a,b). Then f is continuous except possibly at a countable number of points in (a,b).

Proof :- Assume f is increasing.

Furthermore, assume $f(a,b)$ is bounded and f is increasing on $[a,b]$.

Otherwise, express (a,b) as a union of ascending sequence of open intervals the closure of which is contained in (a,b).

Take the union of the discontinuities in each of this countable collection of intervals.

For each $x_0 \in (a,b)$, f has a limit from the left and from the right at x_0.

Define $f(x_0^-) = \lim_{x \to x_0^-} f(x)$

$$= \sup \{ f(x) \mid a < x < x_0 \}$$

$$f(x_0^+) = \lim_{x \to x_0^+} f(x)$$

$$= \inf \{ f(x) \mid x_0 < x < b \}$$

Since f is increasing, $f(x_0^-) \leq f(x_0^+)$.

The function fails to be continuous at x_0 iff $f(x_0^-) < f(x_0^+)$.

Define jump intervals $J(x_0)$ by

$$J(x_0) = \{ y \mid f(x_0^-) < y < f(x_0^+) \}$$

Each jump intervals is contained in

$[f(a), f(b)]$ as the jump intervals are disjoint.

$\therefore$ For each natural number n, there are only a finite number of jump intervals of length greater than $\frac{1}{n}$.

$\therefore$ the set of points of discontinuity is the union of a countable collection of finite sets and therefore is countable.

Thm: Let C be a countable subset of (a,b).
7. Then there is an increasing function on (a,b) that is continuous at the point in $(a,b) \sim C$

Proof: If C is finite then the proof is clear.

Assume C is countably infinite.

Let $\{q_n\}_{n=1}^{\infty}$ be an enumeration of C

Define the function f on (a,b) by setting $f(x) = \sum_{\{n \mid q_n < x\}} \frac{1}{2^n}$ for all $a < x < b$.

f is properly defined, since the geometric series converges.

If $a < u < v < b$, then $f(v) - f(u) = \sum_{\{n \mid u \le q_n < v\}} \frac{1}{2^n} > 0$

$\therefore$ f is increasing.

Let $x_0 = q_k \in C$

$\therefore f(x_0) - f(x) \geq \frac{1}{2^k} \ \forall x < x_0$

$\therefore$ f fails to be continuous at x_0.

Let $x_0 \in (a,b) \sim C$

Let n be a natural number, there is an open interval I containing x_0 for which q_k is not in I; $1 \leq k \leq n$

$\therefore |f(x) - f(x_0)| < \frac{1}{2^n}, \ \forall x \in I$

$\therefore$ f is continuous at x_0.

6.2 <u>Differentiability of monotonic functions:</u>

<u>Defn:</u> A closed bounded interval $[c,d]$ is said to be <u>non-degenerate</u> if <u>$c<d$</u>.

<u>Defn:</u> A collection $\mathcal{F}$ of closed, bounded non-degenerate intervals is said to cover a set E in the sense of Vitali provided for each $x \in E$ and $\varepsilon > 0$ there exists an interval I in $\mathcal{F}$ that contains x and $l(I) < \varepsilon$

The Vitali covering lemma:

Let E be a set of finite outer measure and $\mathcal{F}$ a collection of closed, bounded intervals that covers E in the sense of Vitali. Then for each $\varepsilon > 0$ there is a finite disjoint subcollection $\{I_k\}_{k=1}^{n}$ of $\mathcal{F}$ for which $m^*\left[E \sim \bigcup_{k=1}^{n} I_k\right] < \varepsilon$.

Proof: Since $m^*(E) < \infty$, there is an open set O containing E for which $m(O) < m^*(E) + \infty$

(refer Chapter - 2 Littlewood's principle).

Because $\mathcal{F}$ is a Vitali covering of E, we may assume that each interval in $\mathcal{F}$ is contained in O.

By the countable additivity and monotonicity of measure if $\{I_k\}_{k=1}^{\infty} \subseteq \mathcal{F}$ is disjoint, then,

$$\sum_{k=1}^{\infty} l(I_k) \leq m^*(O) < \infty \rightarrow ①.$$

Since each I_k is bounded and $\mathcal{F}$ is a Vitali covering of E, if $\{I_k\}_{k=1}^{n} \subseteq \mathcal{F}$, then

$E \sim \bigcup_{k=1}^{n} I_k \subseteq \bigcup_{I \in \mathcal{F}_n} \mathcal{F}$ where,

$$\mathcal{F}_n = \{ I \in \mathcal{F} \mid I \cap \bigcup_{k=1}^{n} I_k = \phi \} \to (2)$$

If there is a finite disjoint subcollection of $\mathcal{F}$ that ~~touto~~ covers E, the proof is complete

Otherwise we inductively choose a disjoint countable subcollection $\{I_k\}_{k=1}^{\infty}$ of $\mathcal{F}$ which has the following property

$$E \sim \bigcup_{k=1}^{n} I_k \subseteq \bigcup_{k=n+1}^{\infty} 5 * I_k \;\forall n \to (3).$$

where for a closed bounded interval I, $5*I$ denotes the closed interval that has the same midpoint as I and 5 times its length.

To begin this selection,

Let I_1 be any interval in $\mathcal{F}$. Suppose n is a natural no. and the finite disjoint subcollection $\{I_k\}_{k=1}^{n}$ of $\mathcal{F}$ has been chosen.

Since $E \sim \bigcup_{k=1}^{n} I_k \neq \phi$, the collection $\mathcal{F}_n$ defined in (2) is non-empty.

The supremum, s_n, of the lengths of the intervals in $\mathcal{F}_n$ is finite since

$m^*(O)$ is an upper bound for these lengths.

Choose I_{n+1} to be an interval in $\mathcal{F}_n$ for which $l(I_{n+1}) > \frac{s_n}{2}$.

$\therefore$ This inductively defines $\{I_k\}_{k=1}^{\infty}$, a countable disjoint subcollection of $\mathcal{F}$ such that for each n,

$l(I_{n+1}) > \frac{l(I)}{2}$ if $I \in \mathcal{F}$ & $I \cap \bigcup_{k=1}^{n} I_k = \phi$ $\longrightarrow$ (4)

From (1), $\{l(I_k)\} \to 0$ [$\because$ Let (a_n) be any seq. and $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = l$, if $l>1$, then $(a_n) \to 0$].

Fix a natural no. n,

Now to verify (3).

Let $x \in E \sim \bigcup_{k=1}^{n} I_k$.

From (2), there is an $I \in \mathcal{F}$ which contains x and is disjoint from $\bigcup_{k=1}^{n} I_k$.

Now I must have non-empty intersection with some I_k $(k>n)$, for otherwise from (4), $l(I_k) > \frac{l(I)}{2}$, which contradicts the convergence of $\{l(I_k)\}$ to 0.

Let N be the first natural no, for which $I \cap I_N \neq \phi$. Then $N > n$

Since $I \cap \left(\bigcup_{k=1}^{N-1} I_k\right) = \phi$, from (4), $l(I_N) > \frac{l(I)}{2}$

Since $x \in I$ and $I \cap I_N \neq \phi$, the distance from x to the midpoint of I_N is atmost $l(I) + \frac{l(I_N)}{2}$ ~~since~~.

Since $l(I) < 2l(I_N)$, distance from x to the midpoint of $I_N \leq l(I) + \frac{l(I_N)}{2}$

$$< 2l(I_N) + \frac{l(I_N)}{2}$$

$$= \frac{5}{2} l(I_N)$$

$\Rightarrow x \in 5 * I_N \subseteq \bigcup_{k=n+1}^{\infty} 5 * I_k$.

$\therefore$ (3) is proved.

Let $\varepsilon > 0$ from eqn (1), it follows that there is a natural no. n for which

$\sum_{k=n+1}^{\infty} l(I_k) < \frac{\varepsilon}{5}$

For this choice of n, from (3) and by monotonicity of m^*,

$$m^*\left(E \cap \bigcup_{k=1}^{n} I_k\right) \le m^*\left(\bigcup_{k=n+1}^{\infty} 5 * I_k\right)$$

$$\le \sum_{k=n+1}^{\infty} l(5 * I_k)$$

$$\le 5 \cdot \sum_{k=n+1}^{\infty} l(I_k) < 5 \cdot \frac{\varepsilon}{5}$$

$$= \varepsilon.$$

$$\therefore m^*\left(E \cap \bigcup_{k=1}^{n} I_k\right) < \varepsilon.$$

Defn:- Let f be real valued function and x be an interior point in the domain of f. Then upper derivative of f at x, it is denoted by $\bar{D} f(x)$ and the lower derivative of f at x, $\underline{D} f(x)$ are defined as follows,

$$\bar{D} f(x) = \lim_{h \to 0} \left(\sup_{0<|t|<h} \frac{f(x+t) - f(x)}{t} \right)$$

$$\underline{D} f(x) = \lim_{h \to 0} \left[\inf_{0<|t|<h} \frac{f(x+t) - f(x)}{t} \right]$$

Always $\bar{D} f(x) \ge \underline{D} f(x)$

If $\bar{D} f(x) = \underline{D} f(x)$ and it is finite, we say that f is differentiable at x and we define $f'(x)$ to be the common value of the upper and lower derivatives.

Note: By mean value theorem of Calculus tells us that if a function f is continuous on a closed bounded interval $[c,d]$ and it is differentiable in the interior (c,d) with $f' \geq \alpha$ on (c,d). Then $f(d) - f(c) \geq \alpha(d-c)$

Thm: Let f be an increasing function on the closed bounded interval $[a,b]$. Then for each $\alpha > 0$,

$$m^* \{x \in (a,b) \mid \bar{D} f(x) \geq \alpha\} \leq \frac{1}{\alpha}[f(b) - f(a)]$$

$$\& \ m^* \{x \in (a,b) \mid \bar{D} f(x) = \infty\} = 0.$$

Proof:- Let $\alpha > 0$

Define $E_\alpha = \{x \in (a,b) \mid \bar{D} f(x) \geq \alpha\}$

Choose $\alpha' \in (0,\alpha)$ and $\mathcal{F}$ be the collection of closed bounded intervals $[c,d] \subseteq (a,b)$ and $f(d) - f(c) \geq \alpha'(d-c)$

Since $\bar{D} f \geq \alpha$ on E_α, then $\mathcal{F}$ is a Vitali covering of E_α. Vitali covering lemma tells us that there is a finite disjoint subcollection $\{[c_k, d_k]\}_{k=1}^{n}$

of $\mathcal{I}$ for which $m^*\left[E_\alpha \sim \bigcup_{k=1}^{n}[c_k, d_k]\right] < \varepsilon$.

Since $E_\alpha \subset \bigcup_{k=1}^{n}[c_k, d_k] \cup \left[E_\alpha \sim \bigcup_{k=1}^{n}[c_k, d_k]\right]$

$$m^*(E_\alpha) \le m^*\left[\left(\bigcup_{k=1}^{n}[c_k, d_k]\right] \cup \left(E_k \sim \bigcup_{k=1}^{n}[c_k, d_k]\right)\right)$$

$$\le m^*\left(\bigcup_{k=1}^{n}[c_k, d_k]\right] + m^*\left(E_\alpha \sim \bigcup_{k=1}^{n}[c_k, d_k]\right)$$

$$\le \sum_{k=1}^{n} m^*\left([c_k, d_k]\right) + \varepsilon$$

$$m^*(E_\alpha) \le \sum_{k=1}^{n}(d_k - c_k) + \varepsilon.$$

$$= \sum_{k=1}^{n}\frac{1}{\alpha'}\left[f(d_k) - f(c_k)\right] + \varepsilon.$$

However the function f is increasing on (a,b) and $\{[c_k, d_k]\}_{k=1}^{n}$ is a disjoint collection of subintervals on (a,b)

$$\therefore \sum_{k=1}^{n}\left[f(d_k) - f(c_k)\right] \le f(b) - f(a)$$

$\therefore$ For every $\varepsilon > 0$ and each $\alpha' \in (0, \alpha)$

$$m^*(E_\alpha) \le \frac{1}{\alpha}\left[f(b) - f(a)\right] + \varepsilon$$

For each natural no. n,

$$\{x \in (a,b) \mid \bar{D}f(x) = \infty\} \subseteq \{x \in (a,b) \mid \bar{D}f(x) > n\}$$

$= E_n$

$\therefore m^*\{x \in (a,b) \mid \bar{D} f(x) = \infty\} \leq m^*(E_n)$

$\leq \frac{1}{n}[f(b) - f(a)]$

$\therefore m^*\{x \in (a,b) \mid \bar{D} f(x) = \infty\} = 0$.

<u>Lebesgue theorem</u>:-

If the function f is <u>monotonic</u> on the open interval on (a,b) then it is differentiable a.e on (a,b).

<u>Proof</u>:- Assume f is increasing.

Also assume (a,b) is bounded

Otherwise, express (a,b) as the union of an ascending seq. of open bounded intervals

Let $\{x \in (a,b) \mid \bar{D} f(x) > \underline{D} f(x)\} = \bigcup_{\alpha,\beta} E_{\alpha,\beta}$

where $E_{\alpha,\beta} = \{x \in (a,b) \mid \bar{D} f(x) > \alpha > \beta > \underline{D} f(x)\}$.

<u>T.P</u>:- f is differentiable a.e on (a,b).

<u>Enough to prove</u>:-

$m^*\{x \in (a,b) \mid \bar{D} f(x) > \underline{D} f(x)\} = 0$

Since m^* countably subadditive, we have

$m^*\{x \in (a,b) \mid \bar{D} f(x) > \underline{D} f(x)\} \leq m^*[\bigcup_{\alpha,\beta} E_{\alpha,\beta}]$

$\leq \sum_{\alpha,\beta} m^*[E_{\alpha,\beta}]$

Enough to prove:

For each α, β

$$m^*(E_{\alpha,\beta}) = 0$$

For rationals α, β with $\alpha > \beta$ and set $E = E_{\alpha,\beta}$.

Choose an open set O for which $E \subseteq O \subseteq (a,b)$ and $m(O) < m^*(E) + \epsilon \rightarrow (1)$

Let $\mathcal{F}$ be the collection of all closed, bounded intervals contained in O for which $f(d) - f(c) < \beta(d-c)$

Since $\underline{D}f < \beta$ on E, $\mathcal{F}$ is an Vitali covering of E.

By the Vitali covering lemma, there is a finite disjoint subcollection $\{[c_k, d_k]\}_{k=1}^{n}$ of $\mathcal{F}$ for which

$$m^*\left[E \sim \bigcup_{k=1}^{n} [c_k, d_k]\right] < \epsilon \rightarrow (2)$$

By the choice of the intervals,

$$\bigcup_{k=1}^{n} [c_k, d_k] \subset O.$$

$$\sum_{k=1}^{n} [f(d_k) - f(c_k)] < \beta \left[\sum_{k=1}^{n} (d_k - c_k)\right]$$

$$< \beta\, m(O)$$

$$< \beta(m^*(E) + \epsilon) \text{ by } (1)$$

For $1 \le k \le n$, by applying previous theorem, to the restricted function f to $[c_k, d_k]$.

$$m^*(E \cap [c_k, d_k]) \le \frac{1}{\alpha}[f(d_k) - f(c_k)]$$

$$\therefore m^*(E) = m^*\left[\bigcup_{k=1}^{n}(E \cap [c_k, d_k]) \cup \left(E \sim \bigcup_{k=1}^{n}[c_k, d_k]\right)\right]$$

$$\le \sum_{k=1}^{n} m^*(E \cap [c_k, d_k]) + m^*\left(E \sim \bigcup_{k=1}^{n}[c_k, d_k]\right)$$

$$\le \sum_{k=1}^{n} m^*(E \cap [c_k, d_k]) + \varepsilon \text{ by } (2)$$

$$\le \frac{1}{\alpha}\sum_{k=1}^{n}[f(d_k) - f(c_k)] + \varepsilon.$$

$$\le \frac{\beta}{\alpha}(m^*(E) + \varepsilon) + \varepsilon$$

$$m^*(E) \le \frac{\beta}{\alpha} m^*(E) + \frac{\beta}{\alpha}\varepsilon + \varepsilon$$

$$\left(1 - \frac{\beta}{\alpha}\right) m^*(E) \le \varepsilon' \text{ where } \varepsilon' = \frac{\beta}{\alpha}\varepsilon + \varepsilon$$

Since $0 \le m^*(E) < \infty$, and $\frac{\beta}{\alpha} < 1$, we have $m^*(E) = 0$.

<u>Defn:</u> Let f be integriable over the closed bounded interval $[a, b]$. Extended f to take the value of $f(b)$ on $[b, b+1]$. for $0 < h \le 1$, define the divided difference function

$Diff_h f$ and average valued function $Av_h f(x)$ by $Av_h f(x) = \frac{1}{h} \int_x^{x+h} f \cdot \forall x \in [a,b]$.

& $Diff_h f = \frac{f(x+h) - f(x)}{h}$

Corollary: Let f be an increasing function on the closed bounded interval $[a,b]$. Then f' is integrable over $[a,b]$ and $\int_a^b f' \leq f(b) - f(a)$

Proof: By a result, "A strictly increasing function that is defined on an interval is measurable"

Since f is increasing, f is measurable on $[a, b+1]$. The divided difference function $Diff_h(f)$ is also measurable as $Diff_h f$ is increasing.

By Lebesgue theorem, f is differentiable a.e on (a,b)

$\therefore \{Diff_h f\}$ is a sequence of non-negative measurable functions that converges pointwise a.e on $[a,b]$ to f'.

[Since let $g(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ is defined a.e on $[a,b]$

(i.e) $g(x) = f'(x)$ a.e on $[a,b]$).

$$\lim_{n\to\infty} \text{Diff}_{\frac{1}{n}} f(x) = \lim_{n\to\infty} \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= g(x) \text{ a.e on } [a,b]$$

$$= f'(x) \text{ a.e on } [a,b].$$

According to Fatous lemma,

$$\int_a^b f' \leq \liminf \int_a^b \text{Diff}_{\frac{1}{n}} f(x)\, dx \rightarrow (1)$$

Since f is increasing and for each natural number n,

$$\int_a^b \text{Diff}_{\frac{1}{n}} f.\, dx = \int_a^b \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}} dx$$

$$= n\left[\int_a^b f\left(x+\frac{1}{n}\right) dx - \int_a^b f(x)\, dx\right]$$

Put $x+\frac{1}{n} = y$ & $x = a \Rightarrow y = a+\frac{1}{n}$

$dx = dy$ $\quad x = b \Rightarrow y = b+\frac{1}{n}$.

$$= n\left[\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(y)\, dy - \int_a^b f(y)\, dy\right]$$

$$= n\left[\int_{a+\frac{1}{n}}^{a} f(y)\,dy + \int_{a}^{b} f(y)\,dy + \int_{b}^{b+\frac{1}{n}} f(y)\,dy - \int_{a}^{b} f(y)\,dy\right]$$

$$= n\left[\int_{a+\frac{1}{n}}^{a} f(y)\,dy + \int_{b}^{b+\frac{1}{n}} f(y)\,dy\right]$$

$$= n \cdot f(b) \cdot \frac{1}{n} - n\int_{a}^{a+\frac{1}{n}} f(y)\,dy \qquad [\because f(y) = f(b) \text{ on } y \geq b]$$

$$= f(b) - n\int_{a}^{a+1/n} f(y)\,dy$$

$$\leq f(b) - f(a) \qquad \left[\because x \geq a \to f(x) \geq f(a);\ \int_{a}^{a+1/n} f(x)\,dx \geq \int_{a}^{a+1/n} f(a)\,dx = f(a)\cdot\frac{1}{n}\right]$$

$$\therefore \int_{a}^{b} \mathrm{Diff}_{1/n} f \leq f(b) - f(a)$$

$$\lim_{n\to\infty} \sup \int_{a}^{b} \mathrm{Diff}_{\frac{1}{n}} f \leq f(b) - f(a) \to (2)$$

$$\therefore (1) \,\&\, (2)\ \int_{a}^{b} f' \leq \lim\inf \int_{a}^{b} \mathrm{Diff}_{\frac{1}{n}} f\,dx \leq \lim\sup \int_{a}^{b} \mathrm{Diff}_{\frac{1}{n}} f\,dx \leq f(b) - f(a)$$

$$\therefore \int_{a}^{b} f' \leq f(b) - f(a).$$

6.3 Functions of bounded variation:

Jordan's theorem:

Defn: Let f be a real valued function defined on the closed, bounded interval $[a,b]$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Define the variation of f w.r.t P by $V(f,P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$ and the total variation of f on $[a,b]$ by

$$TV(f) = \sup\{V(f,P) \mid P - \text{a partition of } [a,b]\}.$$

Defn: A real valued function on the closed, bounded interval $[a,b]$ is said to be of bounded variation of $[a,b]$ provided $TV(f) < \infty$ $\boxed{TV(f) < \infty}$

Ex: 1 Let f be an increasing function on $[a,b]$. Then f is bounded variation on $[a,b]$.

For, given ~~by~~ any partition $P = \{x_0, x_1, \ldots, x_n\}$ on $[a,b]$

$$V(f,P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

$$= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = f(b) - f(a)$$

$$V(f,P) = f(b) - f(a) < \infty \quad \forall P$$

$$\Rightarrow TV(f) < \infty$$

$\therefore f$ is of bounded variation on $[a,b]$

Ex: Let f be a Lipschitz function on $[a,b]$. Then f is of bounded variation on $[a,b]$

For, f is a Lipschitz function on $[a,b]$.

$\Rightarrow |f(u) - f(v)| \le c|u-v| \; \forall u,v \in [a,b]$.

Let P be a partition given by

$P = \{x_0, x_1, \dots, x_n\}$.

$$V(f,P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$
$$\le \sum_{i=1}^{n} c|x_i - x_{i-1}|$$
$$\le c \sum_{i=1}^{n} (x_i - x_{i-1})$$
$$\le c(b-a) \text{ where } c>0$$

$\therefore c(b-a)$ is an upper bound for the set of all variations of f with respect to partition of $[a,b]$.

$f(x) = \sqrt{x}$ is continuous on $[0,1]$, but it is not Lipschitz

$\therefore TV(f) \le c(b-a)$.

Hence f is of bounded variation on $[a,b]$.

Ex:- Define the function f on $[0,1]$ by

$$f(x) = \begin{cases} x\cos\left(\frac{\pi}{2x}\right) & 0 < x \le 1 \\ 0 & x = 0 \end{cases}$$

Then f is continuous on $[0,1]$.

For a natural number n,

$P_n = \left\{0, \frac{1}{2n}, \frac{1}{2n-1}, \dots, \frac{1}{3}, \frac{1}{2}, 1\right\}$ of $[0,1]$

Then $V(f, P_n) = \frac{1}{2n} + \frac{1}{2n-2} + \dots + \frac{1}{2}$

$$= \frac{1}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}+\frac{1}{n}\right)$$

Since the harmonic series diverges, f is not of bounded variation on $[0,1]$

Defn:- A partition P of $[a,b]$ that contains the point c induces and is induced by partitions P_1 and P_2 of $[a,c]$ and $[c,b]$ respectively. and for such partition

$$V(f_{[a,b]}, P) = V(f_{[a,c]}, P_1) + V(f_{[c,b]}, P_2)$$

Taking supremum on subpartitions, we conclude that $TV(f_{[a,b]}) = TV(f_{[a,c]}) +$

$$+ TV(f_{[c,b]})$$

Lemma:- Let the function f be of bounded variation on the closed, bounded interval $[a,b]$. Then f has the following explicit expression as the difference of two increasing functions on $[a,b]$:

$$f(x) = \left[f(x) + TV(f_{[a,x]})\right] - TV(f_{[a,x]})$$

Proof:- Consider the function $x \to TV(f_{[a,x]})$ defined on $[a,b]$ is a real valued function.

We call it the total variation function for f.

We know that $TV(f_{[a,b]}) = TV(f_{[a,c]}) + TV(f_{[c,b]})$

For

$a \leq u < v \leq b$, $TV(f_{[a,v]}) = TV(f_{[a,u]}) + TV[f_{[u,v]})$

$\Rightarrow TV(f_{[a,v]}) - TV(f_{[a,u]}) = TV(f_{[u,v]}) \geq 0$

The function $x \to TV(f_{[a,x]})$ is increasing function also for $a \leq u < v \leq b$, take a partition $P = \{u, v\}$ of $[u,v]$. Then

$$f(u) - f(v) \leq |f(v) - f(u)| \leq TV(f_{[u,v]})$$
$$= TV(f_{[a,v]}) - TV(f_{[a,u]})$$

(ie). $f(u) + TV(f_{[a,u]}) \leq f(v) + TV(f_{[a,v]})$

$\therefore x \to f(x) + TV(f_{[a,x]})$ is a real valued increasing function on $[a,b]$.

$$f(x) = (f(x) + TV(f_{[a,x]})) - TV(f_{[a,x]})$$

is the difference of the two increasing functions on $[a,b]$.

Jordan theorem:

A function f is of bounded variation on the closed, bounded interval $[a,b]$ iff it is the difference of two increasing function on $[a,b]$.

Proof: Let f be of bounded variation on $[a,b]$.

Then by the preceeding lemma, f can have an explicit expression as the difference of two increasing function on $[a,b]$.

Conversely, let $f = g-h$ on $[a,b]$ where g and h are increasing function on $[a,b]$.

$$V(f,P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

$$= \sum_{i=1}^{n} |(g(x_i) - g(x_{i-1})) - (h(x_i) - h(x_{i-1}))|$$

$$\le \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| + |h(x_i) - h(x_{i-1})|$$

$$= \sum_{i=1}^{n} (g(x_i) - g(x_{i-1})) + \sum_{i=1}^{n} (h(x_i) - h(x_{i-1}))$$

(since g & h are increasing).

$= (g(b) - g(a)) + (h(b) - h(a))$

Therefore, the set of variations of f with respect to partition of $[a,b]$ is bounded above by $[g(b) - g(a)] + [h(b) - h(a)]$

$\sup\{V(f,P)\}$ exist and it is finite.

$\Rightarrow f$ is of bounded variation of $[a,b]$.

Corollary :- If the function f is of bounded variation on the closed, bounded interval $[a,b]$, then it is differentiable almost everywhere on the (a,b) and f' is integrable over $[a,b]$.

Proof :- According to Jordan's theorem, f is the difference of two increasing functions on $[a,b]$. Therefore, by Lebesgue theorem, f is the difference of two functions which are differentiable a.e on (a,b).

$\therefore f$ is differentiable a.e on (a,b)

$\therefore f'$ is integrable over $[a,b]$ [by previous corollary]

6.4 Absolutely continuous function UNIT-V

Defn: A real valued function f on a closed, bounded interval $[a,b]$ is said to be absolutely continuous on $[a,b]$ provided for each $\varepsilon>0$ there exist a $\delta>0$ such that for every finite disjoint collection $\{(a_k,b_k)\}_{k=1}^{n}$ of open intervals in (a,b) if $\sum_{k=1}^{n}(b_k-a_k)<\delta$ then $\sum_{k=1}^{n}|f(b_k)-f(a_k)|<\varepsilon$ (1)

Note:

1) The criterion for absolutely continuous in the case of a single interval is the criterion for the uniform continuity of f on $[a,b]$.

2) Absolutely continuous $\Rightarrow$ continuous.

3) The converse is not true.

Example The Cantor Lebesgue function is continuous but it is not absolutely continuous.

Theorem: 1 (2)

If the function f is Lipschitz on a ~~[a~~ closed bounded interval $[a,b]$ then it is absolutely continuous on $[a,b]$.

proof: Let $c>0$ be a Lipschitz constant for f on $[a,b]$.

(ie) $|f(u)-f(v)| \leq c|u-v| \ \forall \ u,v \in [a,b]$

By the criterion for absolutely continuous of f

Take $\delta = \frac{\varepsilon}{c}$

then $|u-v| < \delta \Rightarrow |f(u)-f(v)| < \varepsilon$

$\therefore$ f is absolutely continuous on $[a,b]$.

Note:

Define f on $[0,1]$ by $f(x)=\sqrt{x}$, $0 \leq x \leq 1$. is absolutely continuous but not Lipschitz function.

Theorem: 2 (10) Let the function f be absolutely continuous on the closed, bounded interval $[a,b]$ then f is the difference of increasing absolutely continuous functions and in particular, is of bounded variation. (3)

proof:
Claim: f is of bounded variation.

Given f is absolutely continuous on $[a,b]$.

For every $\varepsilon = 1 > 0$, there exists $\delta > 0$.

Let P be a partition of $[a,b]$ into N closed intervals $\{[c_k, d_k]\}_{k=1}^{N}$ each of length less than δ.

Then by the defn δ, in relation to the defn of absolutely continuous, total variation of f is $TV(f_{[c_k, d_k]}) \leq 1$ for $1 \leq k \leq N$.

By additivity formula for finite sums,

$$TV(f) = \sum_{k=1}^{N} TV(f_{[c_k, d_k]}) \leq N.$$

$\therefore f$ is of bounded variation on $[a,b]$

Claim: Total variation of f is absolutely continuous.

Let $\varepsilon > 0$ be given.

Choose $\delta > 0$ as a response to $\varepsilon/2$.
Challenge regarding the criterion for absolutely continuous on $[a,b]$

Let $\{(c_k, d_k)\}_{k=1}^{n}$ be a disjoint subcollection of open intervals of (a,b) for which $\sum_{k=1}^{n}(d_k - c_k) < \delta$. For $1 \le k \le n$ let P_k be the partition of $[c_k, d_k]$.

By the choice of δ in relation to the absolutely continuous on $[a,b]$.

$$\sum_{k=1}^{n} TV\left(f_{[c_k, d_k]}\right)^{P_k} < \frac{\varepsilon}{2}.$$

Take the supremum as for $1 \le k \le n$, P_k vary among partitions of $[c_k, d_k]$ to obtain $\sum_{k=1}^{n} TV\left(f_{[c_k, d_k]}\right) < \varepsilon/2 < \varepsilon$.

~~We know that, Total $TV\left(f_{[c_k, d_k]}\right) = TV\left(f_{[a, d_k]}\right)$~~

$$TV\left(f_{[c_k, d_k]}\right) = TV\left(f_{[a, d_k]}\right) - TV\left(f_{[a, c_k]}\right).$$

If $\sum_{k=1}^{n} |d_k - c_k| < \delta \Rightarrow \left| \sum_{k=1}^{n} TV\left(f_{[a, d_k]}\right) - TV\left(f_{[a, c_k]}\right) \right| < \varepsilon.$

$\therefore$ The total variation function

$x \to TV(f_{[a,x]})$ is absolutely continuous on $[a,b]$

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By theorem,

"Let f be of bounded variation on the closed bounded interval $[a,b]$ then f is the difference of two increasing functions on $[a,b]$"

$$\therefore f(x) = \left[f(x) + TV(f_{[a,x]}) \right] - TV(f_{[a,x]})$$

is the difference of two monotonically increasing functions.

Since the sum of the two absolutely continuous function is absolutely continuous

$f(x) + TV(f_{[a,x]})$ is absolutely continuous

$\therefore$ we obtain, f is the difference of two increasing absolutely continuous functions

Defn: A family $\mathcal{F}$ of measurable function on E is said to be uniformly integrable over E provided for each $\varepsilon > 0$, there is a $\delta > 0$ s.t for each $f \in \mathcal{F}$ if $A \subseteq E$ is measurable and $m(A) < \delta$ then $\int_A |f| < \varepsilon$.

Theorem 3

Let the function f be continuous on the closed, bounded interval $[a,b]$. Then f is absolutely continuous on $[a,b]$ iff the family of divided difference function $\{Diff_h f\}_{0<h\leq 1}$ is uniformly integrable over $[a,b]$

proof. First assume $\{Diff_h f\}_{0<h\leq 1}$ is uniformly integrable over $[a,b]$.

Let $\varepsilon>0$, choose $\delta>0$ for which

$$\int_E |Diff_h f| < \varepsilon/2 \text{ if } m(E)<\delta \text{ and } 0<h\leq 1.$$

claim: δ response to the ε challenge regarding the criterion for f to be absolutely continuous.

Let $\{(c_k, d_k)\}_{k=1}^{n}$ be a disjoint collection of open subintervals of (a,b) for which

$$\sum_{k=1}^{n} [d_k - c_k] < \delta$$

For $0<h\leq 1$, $1\leq k\leq n$

we know that

$$Av_h f(d_k) - Av_h f(c_k) = \int_{c_k}^{d_k} Diff_h f$$

$$\Rightarrow \sum_{k=1}^{n} |Av_h f(d_k) - Av_h f(c_k)| \le \sum_{k=1}^{n} \int_{c_k}^{d_k} |Diff_h f|$$

(7)

$$= \int_E |Diff_h f|$$

where $E = \bigcup_{k=1}^{n} (c_k, d_k)$ & $m(E) < \delta$

By the choice of δ,

$\int_E |Diff_h f| < \varepsilon/2$ (as $\{Diff_h f\}$ is uniform integrable)

$$\Rightarrow \sum_{k=1}^{n} |Av_h f(d_k) - Av_h f(c_k)| < \varepsilon/2$$

Since f is continuous, take the limit $h \to 0^+$ to obtain

$$\sum_{k=1}^{n} |f(d_k) - f(c_k)| \le \varepsilon/2 < \varepsilon$$

$\therefore f$ is absolutely continuous on $[a,b]$.

Conversely, Suppose f is absolutely continuous

$\therefore$ By thm, f is the difference of increasing absolutely continuous functions

Assume that f is increasing function so that the divided difference function is non-negative.

To prove:

Uniform integrability of $\{Diff_h f\}, 0<h\leq 1$

Let $\varepsilon>0$ (i)

To prove: There exists a $\delta>0$ such that for each measurable subset E of (a,b)

$\int_E Diff_h f < \varepsilon$ if $m(E)<\delta$ and $0<h\leq 1$ → (i)

By thm, a measurable set E is contained in a G_δ set G for which $m(G\sim E)=0$

But every G_δ set is the intersection of a decreasing sequence of open sets and every open set is the disjoint union of countable collection of open intervals.

∴ Every open set is the union of an ascending seq of open set, each of which is the union of a finite disjoint collection of open intervals.

By the continuity of integration in order to prove (i).

It is sufficiently to find a $\delta>0$ s.t for $\{[c_k, d_k]\}_{k=1}^n$ a disjoint collection of open intervals of (a,b).

(ii) To prove: $\int_E \text{Diff}_h f < \varepsilon$ if $m(E) < \delta$ 9

where $E = \bigcup_{k=1}^{n} (c_k, d_k)$ and $0 < h \le 1$.

Choose $\delta > 0$ as the response to the $\varepsilon/2$ challenge regarding the criterion for absolutely continuous on $[a, b+1]$.

By the change of variables for Riemann Integration.

$$\int_u^v \text{Diff}_h f = Av_h f(v) - Av_h f(u)$$

$$= \frac{1}{h}\int_v^{v+h} f(x)\,dx - \frac{1}{h}\int_u^{u+h} f(x)\,dx$$

$$= \frac{1}{h}\int_0^h f(v+t)\,dt - \frac{1}{h}\int_0^h f(u+t)\,dt.$$

$$= \frac{1}{h}\int_0^h [f(v+t) - f(u+t)]\,dt$$

$$\int_u^v \text{Diff}_h f = \frac{1}{h}\int_0^h g(t)\,dt \quad \text{where } g(t) = f(v+t) - f(u+t),$$
$0 \le t \le 1$ & $a \le u < v \le b$

$\therefore$ if $\{(c_k, d_k)\}_{k=1}^{n}$ is a disjoint collection of open subintervals of (a,b) & if $E = \bigcup_{k=1}^{n} (c_k, d_k)$

$$\int_E \mathrm{Diff}_h f = \int_{\bigcup_{k=1}^{n}(c_k,d_k)} \mathrm{Diff}_h f$$

(10)

$$= \sum_{k=1}^{n} \int_{c_k}^{d_k} \mathrm{Diff}_h f = \frac{1}{h}\int_0^h g(t)\,dt$$

where $g(t) = \sum_{k=1}^{n}\left[f(d_k+t) - f(c_k+t)\right]$ & $0 \le t \le 1$

If $\sum_{k=1}^{n}[d_k - c_k] < \delta$ then for $0 \le t \le 1$,

$$\sum_{k=1}^{n}\left[(d_k+t) - (c_k+t)\right] < \delta$$

$$\Rightarrow g(t) = \sum_{k=1}^{n}\left[f(d_k+t) - f(c_k+t)\right] < \varepsilon/2$$

$$\int_E \mathrm{Diff}_h f = \frac{1}{h}\int_0^h g(t)\,dt < \varepsilon/2$$

i.e) if $m(E) < \delta \Rightarrow \int_E \mathrm{Diff}_h(f) < \varepsilon/2$

6.5 Integrating derivatives: Differentiating indefinite integral

Let f be a continuous function on a closed, bounded interval $[a,b]$.

Now, $\int_a^b \mathrm{Diff}_h(f) = \mathrm{Av}_h f(b) - \mathrm{Av}_h f(a)$

Theorem: 4

Let the function f be absolutely continuous on the closed, bounded interval $[a,b]$. Then f is differentiable a.e on (a,b), its derivative f' is integrable over $[a,b]$ & $\int_a^b f' = f(b)-f(a)$

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proof: W.K.T,

$$\int_a^b Diff_h f = Av_h f(b) - Av_h f(a)$$

Let $h = 1/n$

Taking limit $h \to 0^+$ then $n \to \infty$.

$$\therefore \lim_{n\to\infty} \int_a^b Diff_{1/n} f = \lim_{n\to\infty} (Av_{1/n} f(b) - Av_{\frac{1}{n}} f(a))$$

$$= f(b) - f(a) \quad -①$$

By thm, f is the difference of increasing function on $[a,b]$

By Lebesgue theorem,

$\therefore f$ is differentiable a.e on $[a,b]$.

$$\therefore \lim_{n\to\infty} Diff_{1/n} f = \lim_{n\to\infty} \frac{f(x+1/n)-f(x)}{1/n}$$

$$= \lim_{n\to\infty} \frac{f(x+h)-f(x)}{h}$$

$$= f'(x) \text{ a.e on } (a,b)$$

$\therefore \{Diff_{1/n} f\}$ converges pointwise a.e to f' on (a,b).

Since f is absolutely continuous on $[a,b]$ by thm

$\{Diff_{1/n} f\}$ is uniformly integrable over $[a,b]$

$\therefore$ By Vitali ~~covering~~ thm, we have

$$\lim_{n\to\infty} \int_a^b Diff_{1/n} f = \int_a^b \lim_{n\to\infty} Diff_{1/n} f = \int_a^b f' \quad \text{——} \quad (2)$$

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From (1) & (2),

$$\int_a^b f' = f(b) - f(a)$$

Defn: A function f on a closed bounded interval $[a,b]$ is the indefinite integral of g over $[a,b]$ provided g is Lebesgue integrable over $[a,b]$ and

$$f(x) = f(a) + \int_a^x g \quad \forall\, x \in [a,b]$$

Theorem: A function f on a closed, bounded interval is absolutely continuous on $[a,b]$ iff it is an indefinite integral over $[a,b]$

proof: Suppose f is absolutely continuous on $[a,b]$.

For each $x \in [a,b]$, f is absolutely continuous on $[a,x]$.

$\therefore$ By the previous thm, f on $[a,x]$, we have

$$f(x) = f(a) + \int_a^x f'$$

$\therefore f$ is the indefinite integral of f' over [a, b].

Conversely, Suppose that f is the indefinite integral of g over $[a, b]$

For a disjoint collection $\{(a_k, b_k)\}_{k=1}^{n}$ of open intervals in (a, b)

Define $E = \bigcup_{k=1}^{n} (a_k, b_k)$

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| = \sum_{k=1}^{n} \left| \int_{a_k}^{b_k} g \right| \quad (\because f \text{ is indefinite integral of } g \text{ over } [a, b])$$

$$\leq \sum_{k=1}^{n} \int_{a_k}^{b_k} |g|$$

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$$= \int_{\bigcup_{k=1}^{n} (a_k, b_k)} |g|$$

$$= \int_E |g|.$$

Let $\varepsilon > 0$ be given

By the ~~Propert~~ Property $2\frac{3}{2}$ (pg. 92),

"Let f be measurable function on E. If f is integrable over E. Then for each $\varepsilon > 0$ there exists $\delta > 0$ for which if $A \subset E$ is measurable and $m(A) < \delta$ then $\int_A |f| < \varepsilon$. we get

Since $|g|$ is integrable over $[a, b]$

By the above result,
there exist $\delta > 0$ s.t. $\int_E |f| < \varepsilon$ if
$E \subset [a,b]$ is measurable and $m(E) < \delta$.

Clearly $E = \bigcup_{k=1}^{n} (a_k, b_k)$ is measurable & (14)

$$m(E) = m\left(\bigcup_{k=1}^{n} (a_k, b_k)\right)$$
$$= \sum_{k=1}^{n} m(a_k, b_k) \leq \sum_{k=1}^{n} (b_k - a_k).$$

Now $m(E) < \delta \Rightarrow \sum (b_k - a_k) < \delta$.

Now, $\int_E |g| < \varepsilon \Rightarrow \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$.

$\therefore f$ is absolutely continuous on $[a,b]$.

<u>Corollary:</u> Let the function f be monotone on a closed bounded interval $[a,b]$.
Then f is absolutely continuous on $[a,b]$
iff $\int_a^b f' = f(b) - f(a)$.

<u>proof:</u> Suppose f is absolutely continuous on $[a,b]$.
By thm, we have $\int_a^b f' = f(b) - f(a)$.

Conversely, f is increasing and
$\int_a^b f' = f(b) - f(a)$

To prove: f is absolutely continuous on [a,b]

By thm, it is enough to prove that f is an indefinite integral over [a,b].

Let $x \in [a,b]$

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Now, $\int_{[a,b]} f' - f(b) - f(a) = 0$

$\int_{[a,x]} f' + \int_{[x,b]} f' - (f(b) - f(a)) = 0$

adding x sub $f(x)$

$\int_a^x f' - (f(x) - f(a)) + \int_x^b f' - (f(b) - f(x)) = 0$.

(by cor thm 4)

(i.e) $\int_a^x f' - (f(x) - f(a)) \leq 0$

& $\int_x^b f' - (f(b) - f(x)) \leq 0$.

Since the sum of two negative no, is zero. Then both are zero

(i.e) $\int_a^x f' - (f(x) - f(a)) = 0$

(i.e) $f(x) = f(a) + \int_a^x f'$,

(i.e) f is the indefinite integral of f' over $[a,b]$

$\therefore$ f is absolutely continuous.

Lemma: Let f be integrable over the closed, bounded interval $[a,b]$. Then $f(x)=0$ for almost all $x \in [a,b]$ iff $\int_{x_1}^{x_2} f = 0$ $\forall (x_1,x_2) \subset [a,b]$

proof: Let f be integrable over $[a,b]$.

Given $f(x)=0$ for almost all $x \in [a,b]$ (1b)

$$\Rightarrow \int_{x_1}^{x_2} f = 0 \quad \forall (x_1,x_2) \subset [a,b]$$

Conversely, Suppose $\int_{x_1}^{x_2} f = 0$ $\forall (x_1,x_2) \subset [a,b]$

claim: $\int_E f = 0$ $\forall$ measurable subset $E \subseteq [a,b]$

Let O be an open set

$O = \bigcup_{n=1}^{\infty} I_n$, $\{I_n\}$ is disjoint collection of open intervals in (a,b).

Now, $\int_O f = \int_{\bigcup_{n=1}^{\infty} I_n} f = \sum_{n=1}^{\infty} \int_{I_n} f = 0$ $\left[\because \int_I f = 0\right]$

Let G be G_δ set

Then G is the intersection of a countable descending collection of open sets.

W.L.G, $G = \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} I_k\right)$

Now, $\int_G f = \int_{\bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} I_k\right)} f = \lim_{n\to\infty} \int_{\bigcup_{k=n}^{\infty} I_k} f = \lim_{n\to\infty} \sum_{k=n}^{\infty} \int_{I_k} f$.

$\therefore \int_G f = 0 \;\forall\; G_\delta$ set G. (17)

By proposition, there exists measurable set $E \subset G$ s.t $E_0 = G \sim E$ & $m^*(E_0) = 0$

Then $E = G \sim (G \sim E)$

$E = G \sim E_0$

$\int_G f = 0 \Rightarrow \int_E f + \int_{E_0} f = 0$

$\therefore \int_E f = 0 \quad [\because \int_{E_0} f = 0$ as $m(E_0) = 0]$

Hence the claim

Define $E^+ = \{x \in [a,b] \mid f(x) \geq 0\}$

$E^- = \{x \in [a,b] / f(x) \leq 0\}$

Then E^+ and E^- are measurable subsets of $[a,b]$

let $f = f^+ - f^-$ where $f^+, f^- \geq 0$.

$\int_a^b f^+ = \int_{E^+} f = 0 \longrightarrow$ by ① and

$\int_a^b -f^- = \int_{E^-} f = 0$ (by ①)

$\therefore f^+ = 0$ & $f^- = 0$ a.e on $[a,b]$ $\left[\because \int_E f = 0 \text{ then } f = 0 \text{ a.e on } E\right]$

$\therefore f = 0$ a.e on $[a,b]$

Thm: 8 (Imp)

Let f be integrable over closed, bounded interval $[a,b]$. Then $\frac{d}{dx}\left(\int_a^x f\right) = f(x)$ for almost all $x \in (a,b)$ (18)

Proof: Define the function F on $[a,b]$ by

$$F[x] = \int_a^x f \quad \forall \; x \in [a,b]$$

Then F is an indefinite integral of f

$\therefore$ By thm, F is differentiable a.e on (a,b) and its derivative F' is integrable

T.P: $\frac{d}{dx}\left(\int_a^x f\right) = f(x)$ a.e on (a,b)

i.e) t.p $F'(x) = f(x)$ a.e on (a,b)

i.e) T.P $F' - f = 0$ a.e on (a,b)

clearly $F' - f = 0$ is integrable over (a,b) $[\because F'$ & f are integrable$]$

Let $[x_1, x_2] \subset [a,b]$ then $\int_{x_1}^{x_2} (F' - f) = \int_{x_1}^{x_2} F' - \int_{x_1}^{x_2} f$

$$= F(x_2) - F(x_1) - \int_{x_1}^{x_2} f$$

$$= \int_{x_1}^{x_2} f - \int_a^{x_1} f - \int_{x_1}^{x_2} f$$

$$= \int_a^{x_2} f - \left(\int_a^{x_1} f + \int_{x_1}^{x_2} f\right)$$

$$= \int_a^{x_2} f - \int_a^{x_2} f$$

$$= 0$$

$$\therefore \int_{x_1}^{x_2} (F'-f) = 0.$$

$\therefore$ By lemma, $F'-f=0$ a.e on $[a,b]$

$\Rightarrow F'=f$ a.e on $[a,b]$.

(i.e) $\frac{d}{dx}\left(\int_a^x f\right) = f(x)$ a.e on $[a,b]$

Defn: A function of bounded variation is said to be singular provided its derivative vanishes a.e

III internal

chapter - 17

General Measure Space their properties and Constructions. (19)

17.1 Measure & Measurable Sets

1. A σ-algebra of subsets of a set X is a collection of subsets of X that contains ϕ and it is closed w.r.t the formation of complements in X and with respect to formation of countable union and by De Morgan's identity w.r.t the formation of intersection.

2) By a set function, μ, we mean a function that assigns a extended real number to certain sets.

Defn:

By a measurable space, we mean a couple (X, m) consisting of a set X and a σ-algebra m of subsets of X. A subset E of X is called measurable (or measurable w.r.t m) provided $E \subseteq M$

Defn:

By a measure μ on a measurable space (X, m), we mean an extended real valued non-negative set function $\mu : m \to [0, \infty)$ for which $\mu(\phi) = 0$ & which is countably additive in the sense that for any countable disjoint collection $\{E_k\}_{k=1}^{n}$ of measurable sets

$$\mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k)$$

By a measure space (X, m, μ) we mean a measurable space (X, m) together with a measure μ defined on m.

Ex: 1 (R, L, m) where R is the set of all real numbers L, the collection of Lebesgue measurable sets and m Lebesgue measure is a measure space.

Ex:2 $(\mathbb{R}, \mathcal{B}, m)$ is a measure space where $\mathcal{B}$ - the collection of all Borel sets m - Lebesgue measure.

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Ex:3 Let X be a set and
$\mathcal{m} = 2^X = \{$collection of all subsets of $X\}$

$$\eta(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite} \end{cases}$$

we call η the ~~continuity~~ counting measure of X and $(X, \mathcal{m}, \eta)$ is a measure space

Ex:4 Let X be a set $x_0 \in X$, $\mathcal{m}$ - a σ-algebra of subsets of X.

Define $\delta_{x_0}(E) = \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{if } x_0 \notin E. \end{cases}$

Then δ_{x_0} is called a Dirac measure

$(X, \mathcal{m}, \delta_{x_0})$ is a Dirac measure space

Ex:5 Let X be any uncountable set & the collection of all those subsets of X that are either countable or a complement of it countable set;

Define $\mu(A) = \begin{cases} 0 & \text{if } A \text{ is a countable set} \\ 1 & \text{if } A^c \text{ is countable} \end{cases}$

$\therefore (X, \mathcal{G}, \mu)$ is a measure space

<u>Theorem</u>: Let (X, m, μ) be a measure space

1. <u>Finite additivity</u>

For any finite disjoint collection $\{E_k\}_{k=1}^{n}$ of measurable sets

$$\mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k).$$

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2) <u>Monotonicity</u>:

If A and B are measurable sets and $A \subset B$ then $\mu(A) \leq \mu(B)$

3) <u>Excision property</u>

If $A \subseteq B$ and $\mu(A) < \infty$, then $\mu(B \sim A) = \mu(B) - \mu(A)$ so that if $\mu(A) = 0$ then $\mu(B \sim A) = \mu(B)$

4) <u>Countable monotonicity</u>

For any countable collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets that <u>covers</u> a measurable set E then

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k).$$

proof: 1) Since μ is countably additive, by setting $E_k = \phi \ \forall \ k > n$

$\Rightarrow \mu(E_k) = 0 \ \forall \ k > n.$

Therefore the finite additive, follows.

2) If $A \subset B$ the $\mu(B) = \mu(A) + \mu(B \sim A)$

$\mu(A) \leq \mu(B)$ (as $\mu(B \sim A) \geq 0$)

3) If $\mu(A) < \infty$ from ①,

$\mu(B \sim A) = \mu(B) - \mu(A)$

$\therefore$ Excision property exists

Further if $\mu(A) = 0$ then $\mu(B \sim A) = \mu(B)$

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4) countable monotonicity:

Define $G_1 = E_1$ & then

define $G_k = E_k \sim \bigcup_{j=1}^{k-1} E_j \quad \forall \ k \geq 2.$

$\therefore \{G_k\}_{k=1}^{\infty}$ is disjoint

Also $\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} G_k$ & $G_k \subset E_k \ \forall k$

$$E \subset \bigcup_{k=1}^{\infty} E_k \Rightarrow \mu(E) \leq \mu\left(\bigcup_{k=1}^{\infty} E_k\right)$$

$$= \mu\left(\bigcup_{k=1}^{\infty} G_k\right)$$

$$= \sum_{k=1}^{\infty} \mu(G_k)$$

i.e.) $\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k)$

Defn: A sequence of sets $\{E_k\}_{k=1}^{\infty}$ is called ascending if for each k, $E_k \subset E_{k+1}$ and said to be descending provided for each k, $E_{k+1} \subset E_k$.

Theorem 9 (Continuity of measure)

Let (X, m, μ) be a measure space

1) If $\{A_k\}_{k=1}^{\infty}$ is an ascending sequence of measurable sets then $\mu\left(\bigcup_{k=1}^{n} A_k\right) = \lim_{k\to\infty} \mu(A_k)$

2) If $\{B_k\}_{k=1}^{\infty}$ is a descending sequence of measurable sets for which $\mu(B_1) < \infty$, then $\mu\left(\bigcap_{k=1}^{n} B_k\right) = \lim_{k\to\infty} \mu(B_k)$

Defn: For a measure space (X, m, μ) and a measurable subset E of X. we say that a property holds a.e on E (or) it holds for almost all x in E, provided it holds on $E \sim E_0$ where E_0 is a measurable subset of E for which $\mu(E_0) = 0$.

Theorem 10 Borel- Cantelli Lemma

Let (X, m, μ) be a measure space and $\{E_k\}_{k=1}^{\infty}$ a countable collection of measurable sets for which $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Then almost all x in X belongs to atmost a finite number of E_k's.

(Same proof above)

Defn 7: Let (X, m, μ) be a measure space the measure μ is called finite provided $\mu(X) < \infty$. It is called a σ-finite provided X is the union of a countable collection of measurable sets each of which has finite measure

Set A measurable set E is said to be of finite measure $\mu(E) < \infty$ and said to be σ-finite provided E is the union of a countable collection of measurable sets each of which has finite measure.

Ex: The counting measure on an uncountable measurable set is not σ-finite.

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Defn: For σ-finite measure, a countable cover by the sets of finite measure may be taken to be disjoint. If $\{X_k\}_{k=1}^{\infty}$ is such a cover for $k \geq 2$ replace X_k by $X_k - \bigcup_{j=1}^{k-1} X_j$ to obtain disjoint cover by the sets of finite measure.

Ex: Lebesgue measure on $R \left(= \bigcup_{n=1}^{\infty} (-n, n) \right)$, uncountable

$\mu(-n, n) = 2n < \infty$. is an example of σ-finite measure

Defn: A measure space (X, m, μ) is said to be Complete provided M contains all subsets of sets of measure zero that is if E belongs to m and $\mu(E) = 0$, then every subset of E also belongs to m.

Ex: Lebesgue measure on the real line is complete.

Note: Lebesgue measure on the real line when restricted to the σ-algebra of Borel sets, is not complete.

Theorem: 11

Let $(X, \mathcal{M}, \mu)$ be a measure space. Define $\mathcal{M}_0$ to be the collection of subsets E of X of the form $E = A \cup B$, where $B \in \mathcal{M}$ & $A \subseteq C$ for some $C \in \mathcal{M}$ for which $\mu(C) = 0$. For such a set E define $\mu_0(E) = \mu(B)$. Then $\mathcal{M}_0$ is a σ-algebra that contains $\mathcal{M}$, μ_0 is a measure that extends μ and $(X, \mathcal{M}_0, \mu_0)$ is a complete measure space. 27

17.2 Signed measure: The Hahn & Jordon Decomposition:

Defn: By a signed measure ν on the measurable space $(X, \mathcal{M})$, we mean an extended real valued set function $\nu : \mathcal{M} \to [-\infty, \infty]$ that posses the following properties

i) ν assume atmost one of the values $+\infty, -\infty$

ii) $\nu(\phi) = 0$.

iii) For any countable collection $\{E_k\}_{k=1}^{\infty}$ of disjoint measurable sets $\nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k)$, where the series $\sum_{k=1}^{\infty} \nu(E_k)$ converges absolutely if $\nu\left(\bigcup_{k=1}^{\infty} E_k\right)$ is finite.

Defn: Let ν be a signed measure. A set A is positive (w.r.t ν) provided A is measurable and for every measurable subset E of A we have $\nu(E) \geq 0$. The restriction of ν to the measurable subsets of a positive set is a measure.

Similarly, a set B is called negative (w.r.t ν) provided it is measurable and every measurable subset of B has non-positive ν measure. The restriction of ν to the measurable subsets of a negative set above is a measure.

A measurable set is called null with respect to ν provided every measurable subset of it has ν measure zero.

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Note: Since a signed measure ν does not take the values ∞ and $-\infty$ for A and B measurable sets if $A \subseteq B$ and $|\nu(B)| < \infty$ then $|\nu(A)| < \infty$.

Theorem: 12

Let ν be a signed measure on the measurable space $(X, \mathcal{M})$ then every measurable subset of a positive set is itself a positive set and the union of countable collection of positive sets is positive

proof: By the ~~addition~~ definition of positive set, every measurable subset of a positive set is itself a positive.

Let $A = \bigcup_{k=1}^{\infty} A_k$ be a countable collection

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of positive sets

Let E be a measurable subset of A

Define $E_1 = E \cap A_1$

For $k \geq 2$ define $E_k = (E \cap A_k) \sim (A_1 \cup A_2 \cup \dots \cup A_{k-1})$

Each E_k is measurable subset of a positive set A_k

ie) $\nu(E_k) \geq 0$, since E is the union of countable disjoint collection $\{E_k\}_{k=1}^{\infty}$

$$\nu(E) = \sum_{k=1}^{\infty} \nu(E_k) \geq 0.$$

$\nu(E) \geq 0$ and A is a positive set.

Hahn's Lemma Thu'13

Let ν be a signed measure on the measurable space (X, M) and E a measurable set for which $0 < \nu(E) < \infty$ then there is a measurable subset A of E that is positive and of positive measure.

proof: If E itself is a positive set, then the proof is complete.

Otherwise, E contains sets of negative measure.

Let m_1 be the smallest natural number for which there is a measurable set of measure $< \frac{-1}{m_1}$.

Choose a measurable set $E_1 \subset E$ with, $\nu(E_1) < \frac{-1}{m_1}$

Let n be a natural no. for which the natural no.s $m_1, m_2, \ldots, m_n$ and measurable sets, $E_1, E_2, \ldots, E_n$ have been chosen such that for $1 \leq k \leq n$; m_k is the smallest natural no for which there is a measurable subset of

$E \sim \bigcup_{j=1}^{k-1} E_j$ for which $\nu(E_k) < \frac{-1}{m_k}$.

If this selection process terminates then the proof is complete.

Otherwise, define $A = E \sim \bigcup_{k=1}^{\infty} E_k$ so that $E = A \cup \left(\bigcup_{k=1}^{\infty} E_k\right)$ is a disjoint decomposition of E.

Since $\bigcup_{k=1}^{\infty} E_k$ is a measurable subset of E

and $|\nu(E)| < \infty$, $\left|\nu\left(\bigcup_{k=1}^{\infty} E_k\right)\right| < \infty$.

By countable additivity of ν,

(31)

$$-\infty < \nu\left(\bigcup_{k=1}^{\infty} E_k\right)$$

$$= \sum_{k=1}^{\infty} \nu(E_k)$$

$$\le \sum_{k=1}^{\infty} \left(\frac{-1}{m_k}\right)$$

Thus $\sum \frac{1}{m_k} < \infty$,

$\therefore \lim_{k \to \infty} m_k = \infty$

Claim that A is a positive set

If B is a measurable subset of A, then for each k, $B \subseteq A \subseteq E \sim \bigcup_{j=1}^{k-1} E_j$

$\therefore$ By the minimal choice of m_k,

$$\nu(B) \geq \frac{-1}{m_k - 1}$$

Since $\lim_{k \to \infty} m_k = \infty$, we have $\nu(B) \geq 0$

$\therefore$ A is a positive set

It remains to prove $\nu(A) > 0$ (32)

$$E = A \cup (E \sim A)$$

$$\nu(E) = \nu(A) + \nu(E \sim A)$$

Now, $\nu(E \sim A) = \nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k) < 0$

& $\nu(E) > 0$

$\therefore \nu(A) > 0$

<u>The Hahn's Decomposition theorem</u>: (L:14)

Let ν be a signed measure on the measurable space $(X, \mathcal{M})$ then there is a positive set A for ν and a negative set B for ν for which $X = A \cup B$, $A \cap B = \phi$.

proof: We assume that $+\infty$ is the infinite value omitted by ν.

Let $\mathcal{P}$ be the collection of positive subsets of X and define $\lambda_1 = \sup\{\nu(E) / E \in P\}$.

Then $\lambda \geq 0$, since P contains an empty set.

Let $\{A_k\}_{k=1}^{\infty}$ be a countable collection of positive sets for which $\lambda = \lim_{k\to\infty} \nu(A_k)$

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Define $A = \bigcup_{k=1}^{\infty} A_k$

$\therefore$ A itself is a positive set

$\Rightarrow \lambda \geq \nu(A)$ — 1

On the other hand, for each k, $A \sim A_k \subset A$

ie) $\nu(A \sim A_k) \geq 0$ ($\because$ A is positive)

Now, $A = A_k \cup (A \sim A_k)$

$\nu(A) = \nu(A_k) + \nu(A \sim A_k)$

$\geq \nu(A_k) \; \forall \, k$ ($\because \nu(A \sim A_k) \geq 0$)

$\therefore \nu(A) \geq \lambda$ ($\because \lambda = \lim_{k\to\infty} \nu(A_k)$) — 2

from 1 & 2 $\therefore \nu(A) = \lambda$ & $\lambda < \infty$. since ν does not take the value ∞.

Let $B = X \sim A$.

<u>To prove:</u> B is negative.

Assume B is not negative

(ie) there is a subset E of B with positive measure

By Hahn's lemma, a subset E_0 of B both positive and of positive measure

(ie) $\nu(E_0) > 0$

(ie) $A \cup E_0$ is a positive set 34

$$\nu(A \cup E_0) = \nu(A) + \nu(E_0)$$
$$> \nu(A) \quad (\because \nu(E_0) > 0)$$
$$= \lambda.$$

$\Rightarrow \Leftarrow$ to the choice of λ.

$\therefore$ B is negative set

<u>Defn:</u> A decomposition of X into the union of disjoint sets A & B for which A is positive for ν and B is negative for ν is called a <u>Hahn Decomposition</u> for ν.

Defn: Two measures ν_1 & ν_2 on (X, m) are said to be mutually singular (in symbols) $(\nu_1 \perp \nu_2)$ if there are disjoint measurable sets A and B, then $X = A \cup B$ with $\nu_1(A) = 0$ and $\nu_2(B) = 0$.

The Jordan Decomposition theorem: 15

Let ν be a signed measure on the measurable space (X, M). Then there are two mutually singular measures ν^+ & ν^- on (X, m) for which $\nu = \nu^+ - \nu^-$. Moreover there is only one such pair of mutually singular measure

Note: $|\nu|(X) = \sup \sum_{k=1}^{n} |\nu(E_k)|$

where the supremum is taken over all finite disjoint collection $\{E_k\}_{k=1}^{n}$ of measurable subsets of X.

For this reason $|\nu|(X)$ is called the total variation of ν and denoted by $\|\nu\|_{var}$

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17.3 The Caratheodory measure induced by an outer measure

Defn: A set function $\mu : S \to [0, \infty]$ defined on a collection S of subsets of a set X is called countably monotone provided whenever a set $E \in S$ is covered by a countable $\{E_k\}_{k=1}^{\infty}$ of sets in S then $\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k)$.

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Note: 1) A measure which is monotone and countably additive, is countably monotone.

2) If the countably monotone set from $\mu : S \to [0, \infty]$ has a property that $\phi \in S$ & $\mu(\phi) = 0$. Then μ is finitely monotone in the sense that whenever $E \in S$ is covered by a finite collection $\{E_k\}_{k=1}^{n}$ of sets in S, then $\mu(E) \leq \sum_{k=1}^{n} \mu(E_k)$.

For, $E_k = \phi \;\; \forall k > n \Rightarrow \mu(E_k) = 0 \;\; \forall k > n$.

Defn: A set function $\mu^*: 2^X \to [0,\infty]$ is called an outer measure provided $\mu^*(\phi)=0$ and μ^* is countably monotone

Defn: For an outer measure $\mu^*: 2^X \to [0,\infty]$. we call a subset E of X measurable with respect to μ^* provided for every subset A of X, $\mu^*(A)=\mu^*(A\cap E)+\mu^*(A\cap E^c)$ $\forall A\subset X$ & $\mu^*(A)<\infty$.

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Note:1 To prove: $E\subset X$ is measurable

Enough to prove: $\mu^*(A)\geq \mu^*(A\cap E)+\mu^*(A\cap E^c)$ $\forall A\subset X$ and $\mu^*(A)<\infty$, since μ^* is finitely monotone.

Note:2 From the defn of measurability, a subset E of X is measurable iff its complement w.r.t X is also measurable.

Note:3 Every set of outer measure zero is measurable.

Given $\mu^*(E)=0$

T.p: E is measurable.

W.K.T, $A \cap E \subset E$.

$0 \leq \mu^*(A \cap E) \leq \mu^*(E) = 0 \Rightarrow \mu^*(A \cap E) = 0$.

Also, $A \cap E^c \subset A$ (38)

$\Rightarrow \mu^*(A \cap E^c) \leq \mu^*(A)$

$\mu^*(A) \geq 0 + \mu^*(A \cap E^c)$

$= \mu^*(A \cap E) + \mu^*(A \cap E^c)$.

$\therefore E$ is measurable

Theorem: 16 Union of a finite collection of measurable sets is measurable.

proof: Claim: union of two measurable sets is measurable

Let $E_1 E_2$ be two measurable sets

Let $A \subset X$, since E_1 is measurable

$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c)$

$= \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap E_1^c \cap E_2^c)$.

Use the identities

$(A \cap E_1^c) \cap E_2^c = A \cap (E_1^c \cap E_2^c) = A \cap (E_1 \cup E_2)^c$.

$(A\cap E_1) \cup (A\cap E_1^c \cap E_2) = A\cap (E_1 \cup E_2)$

$\therefore \mu^*(A) = \mu^*(A\cap E_1) + \mu^*(A\cap E_1^c \cap E_2) + \mu^*(A\cap (E_1 \cup E_2)^c)$

$\geq \mu^*(A\cap (E_1\cup E_2)) + \mu^*(A\cap (E_1 \cup E_2)^c)$. $\forall A \subset X$.

$\therefore E_1 \cup E_2$ is measurable

Let $\{E_k\}_{k=1}^{n}$ be any finite collection of measurable sets.

We prove the measurability of $\bigcup_{k=1}^{n} E_k$ by the method of induction on n.

Suppose it is true for $n-1$ (39)

Now, $\bigcup_{k=1}^{n} E_k = \bigcup_{k=1}^{n-1} E_k \cup E_n$.

Since the union of two measurable sets is measurable, we have

$\bigcup_{k=1}^{n} E_k$ is measurable.

Theorem 4.17 Let $A \subseteq X$, and $\{E_k\}_{k=1}^{n}$ be a finite disjoint collection of measurable sets.

Then $\mu^*(A\cap \bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} \mu^*(A\cap E_k)$. In particular, the restriction of μ^* to the collection of measurable sets is finitely additive.

proof: Proof is by induction on n.
Clearly the result is true for $n=1$.
Assume that it is true for $n-1$.
Since the collection $\{E_k\}_{k=1}^{n}$ is disjoint

(40)

$$A \cap \left(\bigcup_{k=1}^{n} E_k\right) \cap E_n = A \cap E_n \quad \text{— (1)}$$

$$A \cap \left(\bigcup_{k=1}^{n} E_k\right) \cap E_k^c = A \cap \left(\bigcup_{k=1}^{n-1} E_k\right) \quad \text{— (2)}$$

By the measurability of E_n.

$$\mu^*\left(A \cap \left(\bigcup_{k=1}^{n} E_k\right)\right) = \mu^*\left(A \cap \left(\bigcup_{k=1}^{n} E_k\right) \cap E_n\right) + \mu^*\left(A \cap \left(\bigcup_{k=1}^{n} E_k\right) \cap E_n^c\right) \quad \text{by (1) \& (2)}$$

$$= \mu^*(A \cap E_n) + \mu^*\left(A \cap \left(\bigcup_{k=1}^{n-1} E_k\right)\right)$$

$$= \mu^*(A \cap E_n) + \mu^*\left(\bigcup_{k=1}^{n-1} (A \cap E_k)\right)$$

$$= \mu^*(A \cap E_n) + \sum_{k=1}^{n-1} \mu^*(A \cap E_k)$$

$$= \sum_{k=1}^{n} \mu^*(A \cap E_k)$$

Theorem: Union of countable collection of measurable sets is measurable

proof: Let $E = \bigcup_{k=1}^{\infty} E_k$ where each E_k is measurable.

By replacing each E_k with $E_k \sim \bigcup_{i=1}^{k-1} E_i$

we may suppose that $\{E_k\}_{k=1}^{\infty}$ is disjoint

Here each E_k is measurable, since finite union of measurable sets is measurable and a complement in X of a measurable set is measurable.

Let $A \subset X$.

For an index n, define $F_n = \bigcup_{k=1}^{n} E_k$.

Since F_n is measurable and also

$F_n^c \supset E^c \ \forall n$

$$\mu^*(A) = \mu^*(A \cap F_n) + \mu^*(A \cap F_n^c)$$

$$\geq \mu^*(A \cap F_n) + \mu^*(A \cap E^c)$$

By thm,

$$\mu^*(A \cap F_n) = \sum_{k=1}^{n} \mu^*(A \cap E_k)$$

$$\therefore \mu^*(A) \geq \sum_{k=1}^{n} \mu^*(A \cap E_k) + \mu^*(A \cap E^c)$$

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L.H.S of this inequality is independent of n.

$$\therefore \mu^*(A) \geq \sum_{k=1}^{\infty} \mu^*(A \cap E_k) + \mu^*(A \cap E^c)$$

$$\therefore \mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

for all measurable sets A and μ countably monotonic

$$\therefore E = \bigcup_{k=1}^{n} E_k \text{ is measurable}$$

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Thm: 19

Let μ^* be an outer measure on 2^X. Then the collection M of sets that are measurable with respect to μ^* (i.e.) σ-algebra. If $\bar{\mu}$ is a restriction of μ^* on M, then $(X, M, \bar{\mu})$ is a complete measure space.

proof: The complement in X of a measurable subset of X is also measurable.

By thm, the union of countable collection of measurable sets is measurable.

$\therefore$ M is a σ-algebra

$\therefore$ By the defn of outer measure, $\mu^*(\phi) = 0$

As the empty set ϕ is measurable, $\bar{\mu}^*(\phi) = 0$

T.p $\bar{\mu}$ is a measure on M.

Enough to prove that it is countably additive.

Now, μ^* is countably monotone

$\therefore \bar{\mu} = \mu^*/_M$ is countably monotone

$\therefore \bar{\mu}\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \bar{\mu}(E_k)$ 43

Enough to prove:

$\bar{\mu}\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} \bar{\mu}(E_k)$ where $\{E_k\}_{k=1}^{\infty}$ is a disjoint collection of measurable sets

By thm, $\mu^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu^*(E_k)$

Since μ^* is monotone, $\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \mu^*\left(\bigcup_{k=1}^{n} E_k\right)$

$= \sum_{k=1}^{n} \mu^*(E_k) \; \forall n.$

Since L.H.S is independent of n.

$\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} \mu^*(E_k).$

$\therefore \mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu^*(E_k)$

Hence $\bar{\mu} = \mu^*/_M$ is a measure on M.

Let $E \in M$ & $\bar{\mu}(E) = 0$

Let $E' \subset E \Rightarrow 0 \leq \bar{\mu}(E') \leq \bar{\mu}(E) = 0$

$\therefore \bar{\mu}(E') = 0.$

Let A be any set

Then $\mu^*(A \cap E') = 0$ $(\because \mu^*(E') = 0)$

$\mu^*(A) \geq \mu^*(A \cap E'^c)$ $(\because A \cap E'^c \subset A$ & μ^* is monotone$)$

$\mu^*(A) \geq \mu^*(A \cap E'^c) + \mu^*(A \cap E')$

$\therefore E'$ is measurable

Hence $\bar{\mu}$ is complete

$\therefore (X, M, \bar{\mu})$ is a complete measure space

17.4 The construction of outer measure

Theorem: 20 Let S be a collection of subsets of a set X & $\mu: S \to [0, \infty]$ a set function. Define $\mu^*(\phi) = 0$ & for $E \subseteq X$, $E \neq \phi$.

define $\mu^*(E) = \inf \sum_{k=1}^{\infty} \mu(E_k)$ where the infimum is taken over all countable collection $\{E_k\}_{k=1}^{\infty}$ of sets in S that cover E.

Then the set function $\mu^*: 2^X \to [0, \infty]$ is an outer measure called the outer measure induced by μ.

proof: To verify: countable monotonicity

let $\{E_k\}_{k=1}^{\infty}$ be a collection of subsets of X

that covers a set E. If $\mu^*(E_k)=\infty$ for some k, then

$$\mu^*(E) \leq \sum_{k=1}^{\infty} \mu^*(E_k) = \infty.$$

(45)

$\therefore$ We may assume each E_k has finite outer measure.

Let $\varepsilon>0$, for each k, there is a countable collection $\{E_{ik}\}_{i=1}^{\infty}$ of sets in S that covers E_k and

$$\sum_{i=1}^{\infty} \mu(E_{ik}) < \mu^*(E_k) + \frac{\varepsilon}{2^k}.$$

Then $\{E_{ik}\}_{1\leq k, i<\infty}$ is a countable collection of sets in S that covers $\bigcup_{k=1}^{\infty} E_k$ and therefore also covers E.

By the defn of outer measure,

$$\mu^*(E) \leq \sum_{1\leq k,i<\infty} \mu(E_{ik}) = \sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty} \mu(E_{ik})\right]$$

$$\leq \sum_{k=1}^{\infty} \mu^*(E_k) + \sum_{k=1}^{\infty} \varepsilon/2^k$$

$$= \sum_{k=1}^{\infty} \mu^*(E_k) + \varepsilon.$$

Since it holds $\forall \varepsilon>0$,

$$\mu^*(E) \leq \sum_{k=1}^{\infty} \mu^*(E_k).$$

<u>Defn:</u> Let S be a collection of subsets of X, $\mu : S \to [0,\infty]$ a set function and μ^* the outer measure induced by μ. The measure $\bar{\mu}$, the restriction of μ^* to the σ-algebra M of μ^* measurable sets is called the <u>caratheodary measure</u> induced by μ.

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$\mu^* : 2^X \to [0,\infty]$
The induced outer measure

$\mu : S \to [0,\infty]$
a general set function

$\bar{\mu} : M \to [0,\infty]$
The induced caratheodary measure

<u>Ex:</u> S is a collection of open subsets of R then $S_{\sigma\delta}$ is a collection of G_δ subsets of R.

<u>Note:</u> For a collection S of subsets of X, we use S_σ to denote those sets that are countable unions of sets of S and use $S_{\sigma\delta}$ to denote those sets that are countable intersection of sets in S_σ.

Theorem: 21 Fubini & Tonelli - Qn

Let $\mu: S \to [0,\infty]$ be a set function defined on a collection S of subsets of a set X and $\bar{\mu}: M \to [0,\infty]$ the Caratheodory measure induced by μ. Let E be a subset of X for which $\mu^*(E) < \infty$. Then there is a subset A of X for which $A \in S_{\sigma\delta}$, $E \subseteq A$ and $\mu^*(E) = \mu^*(A)$. Furthermore, if E and each set in S is measurable w.r.t μ^*, then so is A and $\bar{\mu}(A \sim E) = 0$.

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proof: Let $\varepsilon > 0$

Claim: There is a set A_ε for which $A_\varepsilon \in S_\sigma$, $E \subseteq A_\varepsilon$ and $\mu^*(A_\varepsilon) < \mu^*(E) + \varepsilon$ — (1).

Since $\mu^*(E) < \infty$, there exists a collection $\{E_k\}_{k=1}^{\infty}$ of sets in S for which $E \subseteq \bigcup_{k=1}^{\infty} E_k$ and $\sum_{k=1}^{\infty} \mu(E_k) < \mu^*(E) + \varepsilon$.

Define $A_\varepsilon = \bigcup_{k=1}^{\infty} E_k$ then $A_\varepsilon \in S_\sigma$, $E \subseteq A_\varepsilon$

Since $\{E_k\}_{k=1}^{\infty}$ is a countable collection of sets in S that covers A_ε, by the defn of outer measure

$\therefore$

$$\mu^*(A_\varepsilon) \le \sum_{k=1}^{\infty} \mu(E_k) < \mu^*(E) + \varepsilon.$$

$$\therefore \mu^*(A_\varepsilon) < \mu^*(E) + \varepsilon.$$

$\therefore$ (1) holds for this choice of $A_\varepsilon \in S_\sigma$.

Define $A = \bigcap_{k=1}^{\infty} A_{1/k}$.

Then $A \in S_{\sigma\delta}$ and $E \subset A$ [$\because E \subset A_{1/k}$ for every choice of k]

$$\Rightarrow \mu^*(E) \le \mu^*(A) \le \mu^*(A_{1/k})$$

$$< \mu^*(E) + 1/k$$

$$\mu^*(A) = \mu^*(E)$$

Assume E is μ^*-measurable set and each set in S is μ^*-measurable set.

Since the collection of measurable sets from a σ-algebra the set A is measurable

But μ^* is an extension of $\bar{\mu}$.

$\therefore$ By the excision property of measure,

$$\bar{\mu}(A \sim E) = \bar{\mu}(A) - \bar{\mu}(E)$$

$$= \mu^*(A) - \mu^*(E) \quad [\because \mu^*(E) = \mu^*(A)]$$

$$\therefore \bar{\mu}(A \sim E) = 0.$$